2018 Math Open At Andover: 10 Sample Individual Round Problems

Andover Math Club

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1 Problems

1. Will is distributing his money to three friends. Since he likes some friends more than others, the amount of money he gives each is in the ratio of $5:3:2$. If the person who received neither the least nor greatest amount of money was given 42 dollars, how many dollars did Will distribute in all?

2. Fan, Zhu, and Ming are driving around a circular track. Fan drives 24 times as fast as Ming and Zhu drives 9 times as fast as Ming. All three drivers start at the same point on the track and keep driving until Fan and Zhu pass Ming at the same time. During this interval, how many laps have Fan and Zhu driven together?

3. Mr. DoBa is playing a game with Gunga the Gorilla. They both agree to think of a positive integer from 1 to 120, inclusive. Let the sum of their numbers be n. Let the remainder of the operation $\frac{n^2}{4}$ be r. If r is 0 or 1, Mr. DoBa wins. Otherwise, Gunga wins. Let the probability that Mr. DoBa wins a given round of this game be p. What is $120p$?

4. Let S be the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. How many subsets of S are there such that if a is the number of even numbers in the subset and b is the number of odd numbers in the subset, then a and b are either both odd or both even? By definition, subsets of S are unordered and only contain distinct elements that belong to S.

5. Phillips Academy has five clusters, WQN, WQS, PKN, FLG and ABB. The Blue Key heads are going to visit all five clusters in some order, except WQS must be visited before WQN. How many total ways can they visit the five clusters?

6. An astronaut is in a spaceship which is a cube of side length 6. He can go outside but has to be within a distance of 3 from the spaceship, as that is the length of the rope that tethers him to the ship. Out of all the possible points he can reach, the surface area of the outer surface can be expressed as $m+n\pi$, where m and n are relatively prime positive integers. What is $m+n$?

7. Let $ABCD$ be a square and E be a point in its interior such that CDE is an equilateral triangle. The circumcircle of CDE intersects sides AD and BC at D, F and C, G, respectively. If $AB = 30$, the area of $AFGB$ can be expressed as $a - b\sqrt{c}$, where a, b, and c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$.

8. Suppose that x, y, z satisfy the equations

$$x + y + z = 3$$
$$x^2 + y^2 + z^2 = 3$$
$$x^3 + y^3 + z^3 = 3$$

   Let the sum of all possible values of x be N. What is $12000N$?

9. In circle O inscribe triangle $\triangle ABC$ so that $AB = 13$, $BC = 14$, and $CA = 15$. Let D be the midpoint of arc BC, and let AD intersect BC at E. Determine the value of $DE \cdot DA$.

10. How many ways are there to color the vertices of a regular octagon in 3 colors such that no two adjacent vertices have the same color?

2 Hints

1. In terms of the ratio, who is the person that received neither the least or the most?
2. When does Fan pass Ming? When does Zhu pass Ming?
3. Test small cases - who wins?
4. a and b are either both odd or both even if and only if $a + b$ is even.
5. How many ways can they do so and visit WQN before WQS?
6. Draw a diagram.
7. Do you see any 30-60-90 triangles?
8. Can you relate these expressions to Vieta's Formulas?
9. Note that $\angle BAD = \angle DAC$.
10. How many ways are there to do so on a triangle? A square?

3 Answers

1. 140
2. 33
3. 120
4. 512
5. 60
6. 360
7. 1203
8. 12000
9. 65
10. 258

4 Solutions

1. The person who received 42 dollars has $\frac{3}{2+3+5}$ths of the money. The total amount of money distributed must be $42 \cdot \frac{10}{3} = \boxed{140}$.

Proposed by William Yue

2. Let Ming's speed in racetrack lengths per hour be m and let the amount of time, in hours, that passes in this race be t. Then we want to find $t(24m + 9m)$, which is the number of racetrack lengths driven by Fan and Zhu.

   Note that the net difference in Fan's speed and Ming's speed is $24m - m = 23m$ and that the net difference in Zhu's speed and Ming's speed is $8m - m = 7m$. Since Fan, Zhu, and Ming meet at the same point at the end of the race, $23mt$ and $7mt$ (the displacements of Fan and Zhu from Ming in racetrack lengths) are both integers. The smallest possible value of mt is therefore $mt = 1$. Our answer is thus $t(24m + 9m) = 33mt = \boxed{33}$.

   Proposed by Justin Chang

3. For any positive integer n, $n^2 \equiv 0, 1 \pmod 4$. Therefore, Mr. DoBa will always win and $120p = 120 \times 1 = \boxed{120}$.

   Proposed by Justin Chang

4. The condition is equivalent to $a + b$ being even, i.e. the number of elements of the subset is an even number. The answer is $\frac{2^{10}}{2} = 2^9 = \boxed{512}$.

   Proposed by Michael Ren

5. Every sequence that satisfies the condition has a complementary sequence in which WQN and WQS switch places. Since the total number of sequences is $5! = 120$, the answer is $120/2 = \boxed{60}$.

   Proposed by Vincent Fan

6. We split the region up into the corners, which put together makes a sphere of radius 3 for an area of 36π, edges, every four of which makes a cylinder side surface of radius 3 and height 6 for an area of $3 \cdot 6\pi \cdot 6 = 108\pi$, and the faces, which contributes the same as the surface area of the spaceship, $6 \cdot 6^2 = 216$. The surface area is $216 + 144\pi$, which leads to an answer of $216 + 144 = \boxed{360}$.

   Proposed by Michael Ren

7. Let H be the foot of the altitude from E to CD. Note that the entire configuration is symmetric about the line EH and that H is the midpoint of CD. Therefore $CDFG$ is a rectangle. Let the center of the circumcircle be O. Since $CDFG$ is symmetric about O, it has 4 times the area of CDO. Using the $30 - 60 - 90$ triangle DOH, we calculate that $OH = \frac{30/2}{\sqrt{3}} = 5\sqrt{3}$. Therefore, $[CDO] = 5\sqrt{3} \cdot 30/2 = 75\sqrt{3}$. $[CDFG] = 300\sqrt{3}$, so $[ABFG] = 900 - 300\sqrt{3}$. Our answer is $\boxed{1203}$.

   Proposed by William Yue

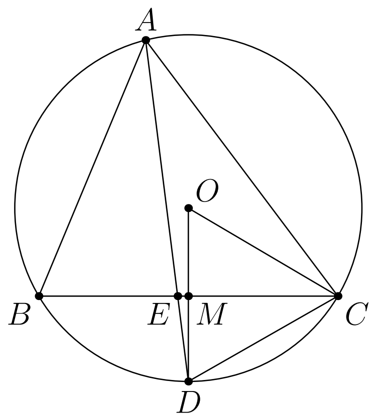
8. We have that

$$xy + yz + zx = \frac{(x+y+z)^2 - (x^2+y^2+z^2)}{2} = 3$$
$$xyz = \frac{(x^3+y^3+z^3) - (x+y+z)^3 - 3(x+y+z)(xy+yz+zx)}{3} = 1$$

   so $(t-x)(t-y)(t-z) = t^3 - (x+y+z)t^2 + (xy+yz+zx)t - xyz = t^3 - 3t^2 + 3t - 1 = (t-1)^3$, implying that the only solution is $x = y = z = 1$. The only possible value for x is 1, so our answer is $12000 \cdot 1 = \boxed{12000}$.

Proposed by Michael Ren

9. Refer to the following diagram:



Note that $\angle BAD = \angle DAC$ because they intercept arcs with equal measure. Therefore, AD is the angle bisector of $\angle BAC$ and $\angle BCD = \angle BAD = \angle DAC$. This means that $\triangle ACD \sim \triangle CED$. We can now write

$$\frac{DA}{DC} = \frac{DC}{DE} \implies DA \cdot DE = DC^2.$$

This result is colloquially known as the *shooting lemma*. The problem reduces to finding the length of DC. To do this, we find the area of $\triangle ABC$ using Heron's formula:

$$[ABC] = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84.$$

The circumradius OC is therefore

$$R = \frac{abc}{4[ABC]} = \frac{13 \cdot 14 \cdot 15}{4 \cdot 84} = \frac{65}{8}.$$

Let M be the midpoint of BC. Then O, M, D are collinear and $\angle OMC = 90^\circ$. Since $MC = \frac{1}{2}BC = 7$, we have

$$OM = \sqrt{OC^2 - MC^2} = \frac{33}{8}.$$

Thus $MD = OD - OM = \frac{65}{8} - \frac{33}{8} = 4$. Finally, $DC^2 = MD^2 + MC^2 = 16 + 49 = \boxed{65}$.

Proposed by Sebastian Zhu

10. Let $f(n)$ be the number of ways to do so to an n-gon. Note that if we set the color of one vertex and keep going around, the first step has 3 choices and each step after that has 2 choices as it cannot be the same color as the previous one. At the end, we either have a valid coloring or an invalid one because the first and last vertices are different colors. In that case, we can fuse the vertices together, obtaining a valid coloring for and $n-1$-gon. Hence, we have the recursion $f(n) = 3 \cdot 2^{n-1} - f(n-1)$. Notably, we have that $f(2) = 6$ as we can color a 2-gon in any way other than both vertices having the same color. This recursion can be rewritten as $f(n) - 2^n = -f(n-1) + 2^{n-1}$, or $g(n) = -g(n-1)$, where $g(n) = f(n) = f(n) - 2^n$. Note that $g(2) = 2$. Then, $f(n) = 2^n + 2(-1)^n$. The answer is then $f(8) = 2^8 + 2 = \boxed{258}$.

Proposed by Michael Ren