2018 MOAA Gunga Bowl: Problems

MOAA 2018 Gunga Bowl Set 1

1. [**5**] Find $1+2+3+4+5+6+7+8+9+10+11$.

2. [**5**] Find $1\cdot 11+2\cdot 10+3\cdot 9+4\cdot 8+5\cdot 7+6\cdot 6$.

3. [**5**] Let $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\frac{1}{6\cdot 7}+\frac{1}{7\cdot 8}+\frac{1}{8\cdot 9}+\frac{1}{9\cdot 10}+\frac{1}{10\cdot 11}=\frac{m}{n}$, where m and n are positive integers that share no prime divisors. Find $m+n$.

MOAA 2018 Gunga Bowl Set 2

4. [**7**] Define $0!=1$ and let $n!=n\cdot(n-1)!$ for all positive integers n. Find the value of $(2!+0!)(1!+8!)$.

5. [**7**] Rachel's favorite number is a positive integer n. She gives Justin three clues about it:

   - n is prime.
   - $n^2-5n+6\neq 0$.
   - n is a divisor of 252.

   What is Rachel's favorite number?

6. [**7**] Shen eats eleven blueberries on Monday. Each day after that, he eats five more blueberries than the day before. For example, Shen eats sixteen blueberries on Tuesday. How many blueberries has Shen eaten in total before he eats on the subsequent Monday?

MOAA 2018 Gunga Bowl Set 3

7. [**9**] Triangle ABC satisfies $AB=7$, $BC=12$, and $CA=13$. If the area of ABC can be expressed in the form $m\sqrt{n}$, where n is not divisible by the square of a prime, then determine $m+n$.

8. [**9**] Sebastian is playing the game *Split!* on a coordinate plane. He begins the game with one token at $(0,0)$. For each move, he is allowed to select a token on any point (x,y) and take it off the plane, replacing it with two tokens, one at $(x+1,y)$, and one at $(x,y+1)$. At the end of the game, for a token on (a,b), it is assigned a score $\frac{1}{2^{a+b}}$. These scores are summed for his total score. Determine the highest total score Sebastian can get in 100 moves.

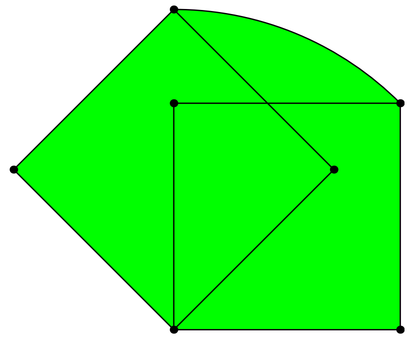
9. [**9**] Find the number of positive integers n satisfying the following two properties:

   - n has either four or five digits, where leading zeros are not permitted,
   - The sum of the digits of n is a multiple of 3.

MOAA 2018 Gunga Bowl Set 4

10. *A unit square rotated* 45° *about a vertex,*
    Sweeps the area for Farmer Khiem's pen.
    If n *is the space the pigs can roam,*
    Determine the floor of $100n$.

    [**11**] If n is the area a unit square sweeps out when rotated 45 degrees about a vertex, determine $\lfloor 100n\rfloor$. Here $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.



11. *Michael is planting four trees,*
In a grid, three rows of three,
If two trees are close,
Then both are bulldozed,
So how many ways can it be?

[**11**] In a three by three grid of squares, determine the number of ways to select four squares such that no two share a side.

12. *Three sixty-seven*
Are the last three digits of
n cubed. What is n?

[**11**] If the last three digits of n^3 are 367 for a positive integer n less than 1000, determine n.

MOAA 2018 Gunga Bowl Set 5

13. [**13**] Determine
$$\sqrt[4]{97+56\sqrt{3}}+\sqrt[4]{97-56\sqrt{3}}.$$

14. [**13**] Triangle $\triangle ABC$ is inscribed in a circle ω of radius 12 so that $\angle B = 68^\circ$ and $\angle C = 64^\circ$. The perpendicular from A to BC intersects ω at D, and the angle bisector of $\angle B$ intersects ω at E. What is the value of DE^2?

15. [**13**] Determine the sum of all positive integers n such that $4n^4 + 1$ is prime.

MOAA 2018 Gunga Bowl Set 6

16. [**15**] Suppose that p, q, r are primes such that $pqr = 11(p + q + r)$ such that $p \geq q \geq r$. Determine the sum of all possible values of p.

17. [**15**] Let the operation $\oplus$ satisfy $a \oplus b = \frac{1}{1/a+1/b}$. Suppose
$$N = (\cdots((2 \oplus 2) \oplus 2) \oplus \cdots 2),$$
where there are 2018 instances of $\oplus$. If N can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers, then determine $m + n$.

18. [**15**] What is the remainder when $\dfrac{2018^{1001}-1}{2017}$ is divided by 2017?

MOAA 2018 Gunga Bowl Set 7

19. [**17**] Let circles ω_1 and ω_2, with centers O_1 and O_2, respectively, intersect at X and Y. A lies on ω_1 and B lies on ω_2 such that AO_1 and BO_2 are both parallel to XY, and A and B lie on the same side of O_1O_2. If $XY = 60$, $\angle XAY = 45^\circ$, and $\angle XBY = 30^\circ$, then the length of AB can be expressed in the form $\sqrt{a - b\sqrt{2} + c\sqrt{3}}$, where a, b, c are positive integers. Determine $a + b + c$.

20. [**17**] If x is a positive real number such that $x^{x^2} = 2^{80}$, find the largest integer not greater than x^3.

21. [**17**] Justin has a bag containing 750 balls, each colored red or blue. Sneaky Sam takes out a random number of balls and replaces them all with green balls. Sam notices that of the balls left in the bag, there are 15 more red balls than blue balls. Justin then takes out 500 of the balls chosen randomly. If E is the expected number of green balls that Justin takes out, determine the greatest integer less than or equal to E.

MOAA 2018 Gunga Bowl Set 8

These three problems are interdependent; each problem statement in this set will use the answers to the other two problems in this set. As such, let the positive integers A, B, C be the answers to problems 22, 23, and 24, respectively, for this set.

22. [**19**] Let $WXYZ$ be a rectangle with $WX = \sqrt{5B}$ and $XY = \sqrt{5C}$. Let the midpoint of XY be M and the midpoint of YZ be N. If XN and WY intersect at P, determine the area of $MPNY$.

23. [**19**] Positive integers x, y, z satisfy

$$\begin{aligned} xy &\equiv A \pmod{5}, \\ yz &\equiv 2A + C \pmod{7}, \\ zx &\equiv C + 3 \pmod{9}. \end{aligned}$$

(Here, writing $a \equiv b \pmod{m}$ is equivalent to writing $m \mid a - b$.)
Given that $3 \nmid x$, $3 \nmid z$, and $9 \mid y$, find the minimum possible value of the product xyz.

24. [**19**] Suppose x and y are real numbers such that

$$\begin{aligned} x + y &= A, \\ xy &= \frac{1}{36}B^2. \end{aligned}$$

Determine $|x - y|$.

MOAA 2018 Gunga Bowl Set 9

25. [**21**] The integer 2017 is a prime which can be uniquely represented as the sum of the squares of two positive integers:

$$9^2 + 44^2 = 2017.$$

If $N = 2017 \cdot 128$ can be uniquely represented as the sum of the squares of two positive integers $a^2 + b^2$, determine $a + b$.

26. [**21**] Chef Celia is planning to unveil her newest creation: a whole-wheat square pyramid filled with maple syrup. She will use a square flatbread with a one meter diagonal and cut out each of the five polygonal faces of the pyramid individually. If each of the triangular faces of the pyramid are to be equilateral triangles, the largest volume of syrup, in cubic meters, that Celia can enclose in her pyramid can be expressed as $\frac{a-\sqrt{b}}{c}$ where a, b and c are the smallest possible possible positive integers. What is $a + b + c$?

27. [**21**] In the Cartesian plane, let ω be the circle centered at $(24, 7)$ with radius 6. Points P, Q, and R are chosen in the plane such that P lies on ω, Q lies on the line $y = x$, and R lies on the x-axis. The minimum possible value of $PQ + QR + RP$ can be expressed in the form $\sqrt{m}$ for some integer m. Find m.

MOAA 2018 Gunga Bowl Set 10

Déjà vu?

28. [**24**] Let ABC be a triangle with incircle ω. Let ω intersect sides BC, CA, AB at D, E, F, respectively. Suppose $AB = 7, BC = 12$, and $CA = 13$. If the area of ABC is K and the area of DEF is $\frac{m}{n} \cdot K$, where m and n are relatively prime positive integers, then compute $m + n$.

   Errata: The problem statement given in the contest incorrectly specified $CD = 13$ instead of $CA = 13$, which is obviously impossible. We awarded 24 points to each team for this question.

29. [**24**] Sebastian is playing the game *Split!* again, but this time in a three dimensional coordinate system. He begins the game with one token at $(0, 0, 0)$. For each move, he is allowed to select a token on any point (x, y, z) and take it off, replacing it with three tokens, one at $(x + 1, y, z)$, one at $(x, y + 1, z)$, and one at $(x, y, z + 1)$ At the end of the game, for a token on (a, b, c), it is assigned a score $\frac{1}{2^{a+b+c}}$. These scores are summed for his total score. If the highest total score Sebastian can get in 100 moves is $\frac{m}{n}$, then determine $m + n$.

30. [**24**] Determine the number of positive 6 digit integers that satisfy the following properties:

   - All six of their digits are $1, 5, 7$, or 8,
   - The sum of all the digits is a multiple of 5.

MOAA 2018 Gunga Bowl Set 11

31. [**27**] The triangular numbers are defined as $T_n = \frac{n(n+1)}{2}$. We also define $S_n = \frac{n(n+2)}{3}$. If the sum

$$\sum_{i=16}^{32} \left(\frac{1}{T_i} + \frac{1}{S_i}\right) = \left(\frac{1}{T_{16}} + \frac{1}{S_{16}}\right) + \left(\frac{1}{T_{17}} + \frac{1}{S_{17}}\right) + \cdots + \left(\frac{1}{T_{32}} + \frac{1}{S_{32}}\right)$$

   can be written in the form $\frac{a}{b}$, where a and b are positive integers with $\gcd(a, b) = 1$, then find $a + b$.

32. [**27**] Farmer Will is considering where to build his house in the Cartesian coordinate plane. He wants to build his house on the line $y = x$, but he also has to minimize his travel time for his daily trip to his barnhouse at $(24, 15)$ and back. From his house, he must first travel to the river at $y = 2$ to fetch water for his animals. Then, he heads for his barnhouse, and promptly leaves for the long strip mall at the line $y = \sqrt{3}x$ for groceries, before heading home. If he decides to build his house at (x_0, y_0) such that the distance he must travel is minimized, x_0 can be written in the form $\frac{a\sqrt{b}-c}{d}$, where a, b, c, d are positive integers, b is not divisible by the square of a prime, and $\gcd(a, c, d) = 1$. Compute $a+b+c+d$.

33. [**27**] Determine the greatest positive integer n such that the following two conditions hold:

   - n^2 is the difference of consecutive perfect cubes;
   - $2n + 287$ is the square of an integer.

MOAA 2018 Gunga Bowl Set 12

The answers to these problems are nonnegative integers **that may exceed 1000000**. You will be awarded points as described in the problems.

34. [**32**] The "Collatz sequence" of a positive integer n is the longest sequence of distinct integers $(x_i)_{i\geq 0}$ with $x_0 = n$ and

$$x_{n+1} = \begin{cases} \frac{x_n}{2} & \text{if } x_n \text{ is even,} \\ 3x_n + 1 & \text{if } x_n \text{ is odd.} \end{cases}$$

   It is conjectured that all Collatz sequences have a finite number of elements, terminating at 1. This has been confirmed via computer program for all numbers up to 2^{64}. There is a unique positive integer $n < 10^9$ such that its Collatz sequence is longer than the Collatz sequence of any other positive integer less than 10^9. What is this integer n?

   An estimate of e gives $\max\{\lfloor 32 - \frac{11}{3}\log_{10}(|n - e| + 1)\rfloor, 0\}$ points.

35. [**32**] We define a graph G as a set $V(G)$ of vertices and a set $E(G)$ of distinct edges connecting those vertices. A graph H is a subgraph of G if the vertex set $V(H)$ is a subset of $V(G)$ and the edge set $E(H)$ is a subset of $E(G)$. Let $ex(k, H)$ denote the maximum number of edges in a graph with k vertices without a subgraph of H. If K_i denotes a complete graph on i vertices, that is, a graph with i vertices and all $\binom{i}{2}$ edges between them present, determine

$$n = \sum_{i=2}^{2018} ex(2018, K_i).$$

An estimate of e gives $\max\{32 - 3\lfloor \log_{10}(|n-e|+1) \rfloor, 0\}$ points.

36. [**32**] Write down an integer between 1 and 100, inclusive. This number will be denoted as n_i, where your Team ID is i. Let S be the set of Team ID's for all teams that submitted an answer to this problem. For every ordered triple of distinct Team ID's (a, b, c) such that $a, b, c \in S$, if all roots of the polynomial

$$x^3 + n_a x^2 + n_b x + n_c$$

are real, then the teams with ID's a, b, c will each receive one virtual banana. If you receive v_b virtual bananas in total and $|S| \geq 3$ teams submit an answer to this problem, you will be awarded

$$\left\lfloor \frac{32 v_b}{3(|S|-1)(|S|-2)} \right\rfloor$$

points for this problem. If $|S| \leq 2$, the team(s) that submitted an answer to this problem will receive 32 points for this problem.