MOAA 2018 Team Round

1. **[40]** In $\triangle ABC$, $AB = 3$, $BC = 5$, and $CA = 6$. Points D and E are chosen such that $ACDE$ is a square which does not overlap with $\triangle ABC$. The length of BD can be expressed in the form $\sqrt{m + n\sqrt{p}}$, where m, n, and p are positive integers and p is not divisible by the square of a prime. Determine the value of $m + n + p$.

2. **[40]** If $x > 0$ and
$$x^2 + \frac{1}{x^2} = 14,$$
find
$$x^5 + \frac{1}{x^5}.$$

3. **[50]** Let BE and CF be altitudes in triangle ABC. If $AE = 24$, $EC = 60$, and $BF = 31$, determine AF.

4. **[50]** Michael and Andrew are playing the game *Bust*, which is played as follows: Michael chooses a positive integer less than or equal to 99, and writes it on the board. Andrew then makes a *move*, which consists of him choosing a positive integer less than or equal to 8 and increasing the integer on the board by the integer he chose. Play then alternates in this manner, with each person making exactly one move, until the integer on the board becomes greater than or equal to 100. The person who made the last move loses. Let S be the sum of all numbers for which Michael could choose initially and win with both people playing optimally. Find S.

5. **[50]** Mr. DoBa likes to listen to music occasionally while he does his math homework. When he listens to classical music, he solves one problem every 3 minutes. When he listens to rap music, however, he only solves one problem every 5 minutes. Mr. DoBa listens to a playlist comprised of 60% classical music and 40% rap music. Each song is exactly 4 minutes long. Suppose that the expected number of problems he solves in an hour does not depend on whether or not Mr. DoBa is listening to music at any given moment, and let m the average number of problems Mr. DoBa solves per minute when he is not listening to music. Determine the value of $1000m$.

6. **[70]** Consider an $m \times n$ grid of unit squares. Let R be the total number of rectangles of any size, and let S be the total number of squares of any size. Assume that the sides of the rectangles and squares are parallel to the sides of the $m \times n$ grid. If $\frac{R}{S} = \frac{759}{50}$, then determine mn.

7. **[70]** For a positive integer k, define the *k-pop* of a positive integer n as the infinite sequence of integers $a_1, a_2, \ldots$ such that $a_1 = n$ and
$$a_{i+1} = \left\lfloor \frac{a_i}{k} \right\rfloor,\ i = 1, 2, \ldots$$
where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Furthermore, define a positive integer m to be *k-pop avoiding* if k does not divide any nonzero term in the k-pop of m. For example, 14 is 3-pop avoiding because 3 does not divide any nonzero term in the 3-pop of 14, which is $14, 4, 1, 0, 0, \ldots$

   Suppose that the number of positive integers less than 13^{2018} which are 13-pop avoiding is equal to N. What is the remainder when N is divided by 1000?

8. **[70]** Suppose that k and x are positive integers such that

$$\frac{k}{2} = \left(\sqrt{1+\frac{\sqrt{3}}{2}}\right)^x + \left(\sqrt{1-\frac{\sqrt{3}}{2}}\right)^x.$$

Find the sum of all possible values of k.

9. **[80]** Quadrilateral $ABCD$ with $AC = 800$ is inscribed in a circle, and E, W, X, Y, Z are the midpoints of segments BD, AB, BC, CD, DA, respectively. If the circumcenters of EWZ and EXY are O_1 and O_2, respectively, determine O_1O_2.

10. **[80]** Vincent is playing a game with Evil Bill. The game uses an infinite number of red balls, an infinite number of green balls, and a very large bag. Vincent first picks two nonnegative integers g and k such that $g < k \leq 2016$, and Evil Bill places g green balls and $2016 - g$ red balls in the bag, so that there is a total of 2016 balls in the bag. Vincent then picks a ball of either color and places it in the bag. Evil Bill then inspects the bag. If the ratio of green balls to total balls in the bag is ever exactly $\frac{k}{2016}$, then Evil Bill wins. If the ratio of green balls to total balls is greater than $\frac{k}{2016}$, then Vincent wins. Otherwise, Vincent and Evil Bill repeat the previous two actions (Vincent picks a ball and Evil Bill inspects the bag). If S is the sum of all possible values of k that Vincent could choose and be able to win, determine the largest prime factor of S.