2018 MOAA Gunga Bowl: Solutions

Andover Math Club

1 Set 1 Solutions

1. We pair up the terms as follows:

   $1+2+3+4+5+6+7+8+9+10+11 = (1+11)+(2+10)+(3+9)+(4+8)+(5+7)+6 = 12\cdot 5+6 = \boxed{66}$.

   Proposed by Justin Chang

2. By the difference of squares formula, $36 - x^2 = (6-x)(6+x)$. Thus,

   $1\cdot 11 + 2\cdot 10 + 3\cdot 9 + 4\cdot 8 + 5\cdot 7 + 6\cdot 6 = 6^2\cdot 6 - 5^2 - 4^2 - 3^2 - 2^2 - 1^2 - 0^2 = 216 - \dfrac{5\cdot 6\cdot 11}{6} = \boxed{161}$.

   Alternatively, the expression can be computed directly:

   $$1\cdot 11 + 2\cdot 10 + 3\cdot 9 + 4\cdot 8 + 5\cdot 7 + 6\cdot 6 = 11 + 20 + 27 + 32 + 35 + 36 = 161.$$

   Proposed by Justin Chang

3. Note that
   $$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}.$$
   We can telescope this sum into
   $$\frac{m}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \cdots - \frac{1}{10} + \frac{1}{10} - \frac{1}{11} = 1 - \frac{1}{11} = \frac{10}{11},$$
   which results in $10 + 11 = \boxed{21}$.

   Proposed by Justin Chang

2 Set 2 Solutions

4. We compute $2! = 2$, $0! = 1$, $1! = 1$, and $8! = 40320$. Then $(2+1)(1+40320) = \boxed{120963}$.

   Proposed by Sebastian Zhu

5. We prime factorize $252 = 2^2 \cdot 3^2 \cdot 7$. The number n is prime, so it must be one of the numbers $2, 3, 7$. The roots of $x^2 - 5x + 6 = (x-2)(x-3)$ are 2 and 3, so we know that n must be $\boxed{7}$.

   Proposed by Justin Chang

6. This is equivalent to summing an arithmetic sequence with first term 11, common difference 5, and last term $11 + 6\cdot 5 = 41$. Therefore the sum is equal to
   $$\frac{n(a_1 + a_n)}{2} = \frac{7(11+41)}{2} = \boxed{182}.$$

   Proposed by Sebastian Zhu

3 Set 3 Solutions

7. We can apply Heron's formula to triangle ABC. We define $a = BC, b = CA$, and $c = AB$. Since its semiperimeter $s = \frac{a+b+c}{2} = 16$, we have

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{16 \cdot 9 \cdot 4 \cdot 3} = 24\sqrt{3},$$

which results in $24 + 3 = \boxed{27}$.

Proposed by William Yue

8. The key observation is that no move changes the score. If Sebastian performs a move on a token at (x, y), then the score after the move equals the score before the move:

$$\frac{1}{2^{x+1+y}} + \frac{1}{2^{x+y+1}} = \frac{2}{2^{x+y+1}} = \frac{1}{2^{x+y}}.$$

Therefore, the answer is equal to the original total score, which is just $\frac{1}{2^{0+0}} = \boxed{1}$.

Proposed by William Yue

9. The second condition is just equivalent to the number being divisible by 3. The least four or five digit number that is a multiple of 3 is 1002, and the greatest is 99999. Thus, our answer is

$$\frac{99999 - 1002}{3} + 1 = \boxed{33000}.$$

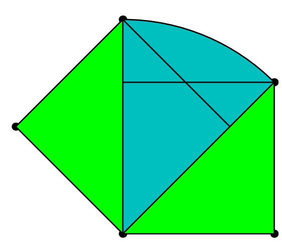
Proposed by William Yue

4 Set 4 Solutions

10. The desired area consists of two 45-45-90 triangles with legs of length one and one 45° sector of a circle with radius $\sqrt{2}$. Therefore, the desired area is

$$n = 2\left(\frac{1}{2} \cdot 1 \cdot 1\right) + \frac{1}{8} \cdot \pi(\sqrt{2})^2 = 1 + \frac{\pi}{4}.$$

Computation reveals that $\lfloor 100 + 25\pi \rfloor = \boxed{178}$.



Proposed by William Yue

11. We have two cases here:

    (a) **There is a tree in the middle square.** As a result, only the four corner squares can have trees planted in them. There are then $\binom{4}{3}$ ways to pick three that work.

    (b) **There is no tree in the middle square.** As a result, the only ways to pick four squares are to choose all of the corner squares or to choose all of the central edge squares.

There are $\binom{4}{3} + 2 = \boxed{6}$ ways in total.

Proposed by William Yue

12. We perform a digit-by-digit extraction as follows. Since $n^3 \equiv 7 \pmod{10}$, we know that $n \equiv 3 \pmod{10}$. Let $n = 10a + 3$. Then

$$367 \equiv n^3 \equiv 1000a^3 + 900a^2 + 270a + 27 \equiv 70a + 27 \pmod{100}.$$

Then

$$70a \equiv 40 \pmod{100} \implies a \equiv 2 \pmod{10}.$$

Let $a = 10b + 2$. Then $n = 10(10b + 2) + 3 = 100b + 23$, so

$$367 \equiv n^3 \equiv 10^6b^3 + 690000b^2 + 158700b + 12167 \equiv 700b + 167 \pmod{1000}.$$

Therefore

$$700b \equiv 200 \pmod{1000} \implies b \equiv 6 \pmod{10}.$$

However, b must be a digit, as otherwise $n = 100b + 23 > 1000$. Thus $b = 6$, and $n = \boxed{623}$.

Proposed by Sebastian Zhu

5 Set 5 Solutions

13. The most straightforward way to solve this problem is to try to find the explicit values of $\sqrt[4]{97 + 56\sqrt{3}}$ and $\sqrt[4]{97 - 56\sqrt{3}}$. Optimistically, we set

$$\begin{aligned}\sqrt[4]{97 + 56\sqrt{3}} &= a + b\sqrt{3} \\ 97 + 56\sqrt{3} &= \left(a + b\sqrt{3}\right)^4 \\ 97 + 56\sqrt{3} &= a^4 + 4\sqrt{3}a^3b + 18a^2b^2 + 12\sqrt{3}ab^3 + 9b^4.\end{aligned}$$

Then, matching rational and irrational parts, we find that $97 = a^4 + 18a^2b^2 + 9b^4$ and $56 = 4a^3b + 12ab^3$. We look at the second equation because it has less terms:

$$\begin{aligned}56 &= 4a^3b + 12ab^3 \\ 14 &= ab(a^2 + 3b^2).\end{aligned}$$

Since we want a and b to be positive integers, we find that $(a, b) = (2, 1)$ satisfies this equation, and in fact it also satisfies the other equation, $97 = a^4 + 18a^2b^2 + 9b^4$! Therefore $\sqrt[4]{97 + 56\sqrt{3}} = 2 + \sqrt{3}$. Similarly, we find that $\sqrt[4]{97 - 56\sqrt{3}} = 2 - \sqrt{3}$, so that

$$\sqrt[4]{97 + 56\sqrt{3}} + \sqrt[4]{97 - 56\sqrt{3}} = 2 + \sqrt{3} + 2 - \sqrt{3} = \boxed{4}.$$

Proposed by William Yue

14. Recall that the measure of an inscribed angle is half that of the central angle. Therefore

$$\angle COE = 2\angle CBE = \angle B = 68^\circ.$$

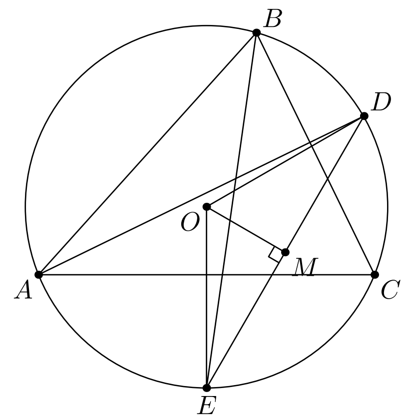
Also, since AD is a height we know that $\angle DAC = 90^\circ - \angle C$. Thus

$$\angle DOC = 2\angle DAC = 180^\circ - 2\angle C = 52^\circ.$$

Now we find

$$\angle DOE = \angle DOC + \angle COE = 68^\circ + 52^\circ = 120^\circ.$$

Now draw the perpendicular from O to DE, and call the point of intersection M. Note that OM bisects $\angle DOE$, so that $\angle MOE = 60^\circ$. This means that $ME = \frac{\sqrt{3}}{2}OE$ since $\triangle OME$ is a $30 - 60 - 90$ right triangle. Finally, $DE = 2ME = \sqrt{3}OE$, so $DE^2 = 3 \cdot OE^2 = \boxed{432}$.



Proposed by Sebastian Zhu

15. We can factor the first term by adding and subtracting $4n^2$;

$$4n^4 + 1 = 4n^4 + 4n^2 + 1 - 4n^2 = (2n^2 + 1)^2 - (2n)^2 = (2n^2 + 2n + 1)(2n^2 - 2n + 1)$$

by difference of squares. This number is only prime if one of its factors is equal to 1. Since $0 < 1-2n+2n^2 < 1+2n+2n^2$, only $1-2n+2n^2$ can equal 1. This factor equals 1 when $2n^2-2n = 0 \Rightarrow 2n(n-1) = 0$, so $n = 1$ is our only answer. The sum of all possible n is $\boxed{1}$.

Proposed by William Yue

6 Set 6 Solutions

16. For the moment, we can ignore the condition $p \geq q \geq r$, as the equation is symmetric. Since 11 divides the right hand side of the equation, it also divides the left. Thus, $pqr = 11$, and since p, q, r are primes, we must have one of them equal to 11. Without loss of generality, allow $p = 11$. Then

$$\begin{aligned} qr &= q + r + 11 \\ qr - q - r + 1 &= 12 \\ (q-1)(r-1) &= 12 \end{aligned}$$

by Simon's Favorite Factoring Trick. This results in three possible cases:

- $q - 1 = 12$ and $r - 1 = 1$, which leads to $q = 13, r = 2$,
- $q - 1 = 6$ and $r - 1 = 2$, which leads to $q = 7, r = 3$,
- $q - 1 = 4$ and $r - 1 = 3$, which leads to $q = 5, r = 4$, which doesn't work as 4 isn't prime.

This results in the two triplets $(p, q, r) = (13, 11, 2)$ and $(11, 7, 3)$, as we must incorporate $p \geq q \geq r$. Thus the sum of all possible values of p is $13 + 11 = \boxed{24}$.

Proposed by Sebastian Zhu

17. Let x_n represent

$$(\cdots((2 \oplus 2) \oplus 2) \oplus \cdots 2),$$

where there are n instances of $\oplus$. We have that,

$$x_{n+1} = \frac{1}{1/2 + 1/x_n}.$$

Since $x_0 = \frac{2}{0+1}$ and $x_1 = \frac{2}{1+1}$, it is safe to assume that $x_n = \frac{2}{n+1}$ for all n. In fact, we can show this by induction. Assuming $x_{n-1} = \frac{2}{n}$, we have

$$x_{n+1} = \frac{1}{1/2 + n/2} = \frac{2}{n+1},$$

as desired. We have already resolved the base case of 0. Therefore, $x_{2018} = \frac{2}{2019}$, which results in $2 + 2019 = \boxed{2021}$.

Proposed by William Yue

18. Write $2018^{1001} - 1$ as

$$(2018 - 1)(2018^{1000} + 2018^{999} + \cdots + 2018 + 1).$$

Then

$$\frac{2018^{1001} - 1}{2017} = 2018^{1000} + 2018^{999} + \cdots + 2018 + 1,$$

which reduces to

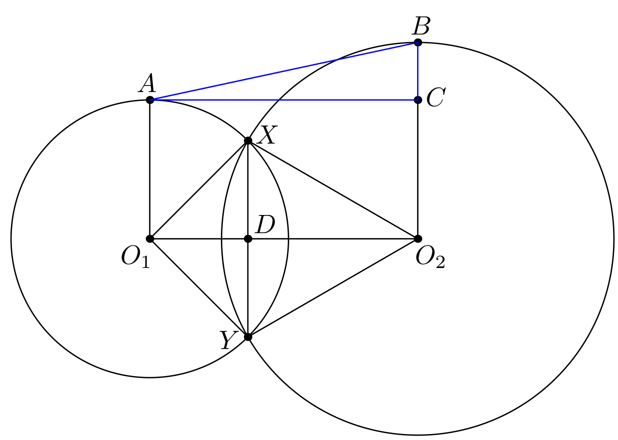
$$1 + 1 + \cdots + 1 + 1 \equiv \boxed{1001} \pmod{2017}.$$

Proposed by Sebastian Zhu

7 Set 7 Solutions

19. Since $\angle XAY = 45°$, the arc XY in circle ω_1 must have length $2 \cdot 45° = 90°$, so $\angle XO_1Y = 90°$. Similarly, $\angle XO_2Y = 90°$. Therefore, as $O_1X = O_1Y$ and $O_2X = O_2Y$, O_1XY is a $45 - 45 - 90$ triangle and O_2XY is equilateral. Therefore, if D is the intersection of O_1O_2 and XY,

$$O_1O_2 = O_1D + O_2D = \frac{1}{2} \cdot XY + \frac{\sqrt{3}}{2} \cdot XY = 30 + 30\sqrt{3}.$$



Now, let C be the foot of the altitude from A onto BO_2. Then, $AC = O_1O_2$ and

$$BC = BO_2 - AO_1 = O_2X - O_1X = 60 - 30\sqrt{2}.$$

Finally, using the Pythagorean Theorem on right triangle ABC gives

$$AB = \sqrt{AC^2 + AB^2} = \sqrt{(30 + 30\sqrt{3})^2 + (60 - 30\sqrt{2})^2} = \sqrt{9000 - 3600\sqrt{2} + 1800\sqrt{3}},$$

which results in $9000 + 3600 + 1800 = \boxed{14400}$.

Proposed by William Yue

20. Square the equation to obtain

$$(x^2)^{x^2} = 2^{160} = 32^{32}.$$

Therefore $x^2 = 32 \implies x^3 = 32\sqrt{32} = 128\sqrt{2}$. Using our computation skills we find that

$$181 < 128\sqrt{2} < 182,$$

so the answer is $\boxed{181}$.

Proposed by Sebastian Zhu

21. Suppose the bag contains b blue balls after Sam replaces some of them. Then, it has $b + 15$ red balls and $735 - 2b$ green balls. The expected number of green balls when selecting 500 balls at random is thus

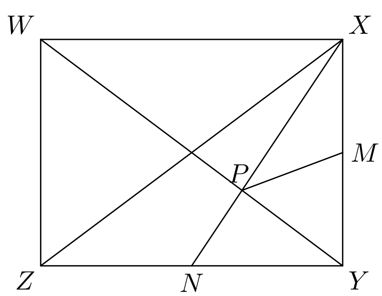
$$\mathbb{E}\left[\frac{500}{750}(735 - 2b)\right] = 490 - \frac{4}{3}\mathbb{E}[b].$$

Note that b ranges over all the values from 1 to 367, each with equal probability (as the number of blue balls fixes the number of red and green ones). Therefore, $\mathbb{E}[b] = 184$, so our answer is $\lfloor 490 - \frac{4}{3} \cdot 184 \rfloor = \boxed{244}$.

Proposed by William Yue

8 Set 8 Solutions

Let's solve the first problem (problem 22) first (in terms of the other answers).



Consider $\triangle XYZ$. We see that XN is a median of this triangle, and that WY is also a median of the triangle because the diagonals of a rectangle bisect each other. Thus P is the centroid of $\triangle XYZ$. Using the fact that the medians of a triangle divide the triangle into six regions of equal area, we see that

$$[MPNY] = \frac{1}{3}[XYZ] = \frac{1}{3} \cdot \frac{1}{2}[WXYZ] = \frac{1}{6}[WXYZ],$$

so $[MPNY] = \frac{1}{6}\sqrt{5B} \cdot \sqrt{5C} = \frac{5}{6}\sqrt{BC}$. We conclude that $A = \frac{5}{6}\sqrt{BC}$, which implies that $A^2 = \frac{25}{36}BC$.

We now turn our attention to the third problem (problem 24). We see that $x + y = A$ implies that

$$\begin{aligned} x^2 + 2xy + y^2 &= A^2 \\ x^2 - 2xy + y^2 &= A^2 - 4xy \\ (x - y)^2 &= A^2 - \frac{1}{9}B^2 \\ |x - y| &= \sqrt{A^2 - \frac{1}{9}B^2}, \end{aligned}$$

which means that $C = \sqrt{A^2 - \frac{1}{9}B^2}$, and therefore $C^2 = A^2 - \frac{1}{9}B^2$. Plugging in $A^2 = \frac{25}{36}BC$ and letting $x = \frac{B}{C}$, we see that

$$\begin{aligned} C^2 &= \frac{25}{36}BC - \frac{1}{9}B^2 \\ 36 &= 25x - 4x^2 \\ 0 &= (4x-9)(x-4). \end{aligned}$$

Therefore $\frac{B}{C} = \frac{9}{4} \implies B = \frac{9}{4}C$ or $\frac{B}{C} = 4 \implies B = 4C$. The first case gives $A^2 = \frac{9}{16}C^2 + C^2 \implies A = \frac{5}{4}C$. If we let $C = 4k$ then we get the solution $(A, B, C) = (5k, 9k, 4k)$. The second case gives $A^2 = \frac{16}{9}C^2 + C^2 \implies A = \frac{5}{3}C$. If we let $C = 3k$ then we get the solution $(A, B, C) = (5k, 12k, 3k)$. Armed with this knowledge, we inspect the second problem (problem 23).

One thing catches our attention in this knowledge: $3 \nmid x, z$. This means that $3 \nmid C + 3 \implies 3 \nmid C$, using the last congruence. Therefore the case $(A, B, C) = (5k, 12k, 3k)$ is ruled out by this condition, because C is always divisible by 3 in this case. Now we know that $(A, B, C) = (5k, 9k, 4k)$. Therefore the first congruence reduces to

$$xy \equiv 5k \equiv 0 \pmod{5}$$

and the second congruence reduces to

$$yz \equiv 14k \equiv 0 \pmod{7}.$$

Multiplying these congruences by z and x respectively yields

$$xyz \equiv 0 \pmod{5}$$

and

$$xyz \equiv 0 \pmod{7}.$$

Finally, multiplying the last congruence by y yields

$$xyz \equiv (C+3)y \equiv 0 \pmod{9},$$

since we know that $9 \mid y$. Therefore $xyz \equiv 0 \pmod{315}$, and since x, y, z are positive integers we know that $xyz \geq 315$. To show that equality can hold is trivial; simply let $(x, y, z) = (5, 9, 7)$. This produces the solution $B = 315$. Then $k = 35$, and we get the solution $\boxed{(A, B, C) = (175, 315, 140)}$. This is the answer; it can be verified to work by simply plugging the numbers back into each of the problems.

Note. We made one huge implicit assumption in the problem, and it was that $(A, B, C) = (5k, 12k, 3k)$ implies that A is divisible by 5, B is divisible by 12, and C is divisible by 3; in other words, we assumed that k was an integer. Why can we do this? All we know is that A, B, C are positive integers, nothing about k. However, we do know that $k = \frac{A}{5}$, so we can indeed conclude that k is rational. Then $\frac{12A}{5} = B$, so it is possible to now conclude that $5 \mid A$. A similar argument proves this for the other cases; in general, this is true whenever all the constants in front of the k are relatively prime when considered together. Therefore, even though 12 and 3 are not relatively prime, we can still conclude that k is an integer because they are relatively prime altogether with 5. This argument works for the case $(A, B, C) = (5k, 9k, 4k)$ as well.

22. $\boxed{175}$

Proposed by William Yue

23. $\boxed{315}$

Proposed by Sebastian Zhu

24. $\boxed{140}$

Proposed by Sebastian Zhu

9 Set 9 Solutions

25. After making the key observation that $2a^2 + 2b^2 = (a+b)^2 + (a-b)^2$, we write

$$\begin{aligned} 2017 \cdot 128 &= (9^2 + 44^2) \cdot 128 \\ &= (35^2 + 53^2) \cdot 64 \\ &= 280^2 + 424^2. \end{aligned}$$

Thus $a + b = \boxed{704}$.

Proposed by Sebastian Zhu

26. The side length of the flatbread is equal to $\frac{1}{\sqrt{2}}$. The volume of enclosed syrup is maximized when the side length of the pyramid is maximized. Let this side length be s. Then by packing the faces like so (see image), we can achieve $s = \frac{\frac{1}{\sqrt{2}}}{1+\frac{\sqrt{3}}{2}} = \frac{\sqrt{2}}{2+\sqrt{3}} = 2\sqrt{2} - \sqrt{6}$. It is not hard to see that the volume of the square pyramid in terms of s is given by $V = \frac{bh}{3} = \frac{s^2}{3}h = \frac{s^3}{3\sqrt{2}}$, so the maximum volume of syrup enclosed is equal to

$V = \frac{s^3}{3\sqrt{2}} = \frac{52-30\sqrt{3}}{3} = \frac{52-\sqrt{2700}}{3}$ and thus $a + b + c = 52 + 2700 + 3 = 2755$.

Proposed by Andy Xu

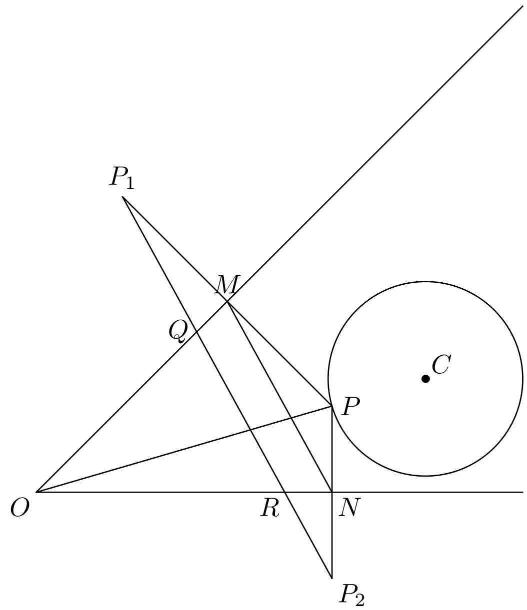
27. Reflect P over $y = x$ and $y = 0$ to get P_1 and P_2, respectively, and let M and N be the midpoints of PP_1 and PP_2, respectively. Then

$$PQ + QR + RP = P_1Q + QR + RP_2 \geq P_1P_2 = 2MN,$$

so it suffices to minimize MN. Now let $P = (x, y)$, so that $M = \left(\frac{x+y}{2}, \frac{x+y}{2}\right)$ and $N = (x, 0)$. Then

$$MN = \sqrt{\left(\frac{x+y}{2} - x\right)^2 + \left(\frac{x+y}{2} - 0\right)^2} = \sqrt{\frac{1}{2}\left(x^2 + y^2\right)} = \frac{\sqrt{2}}{2}OP.$$

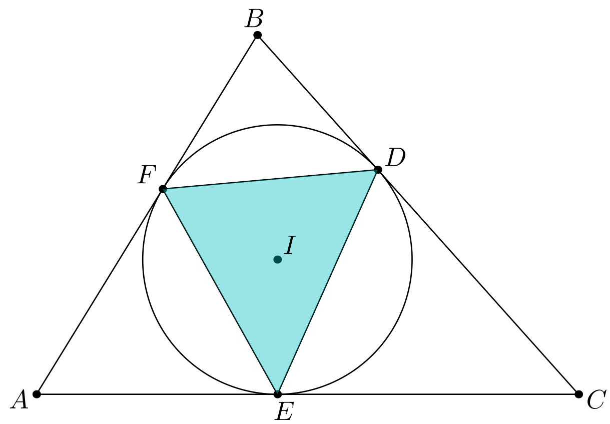
Therefore it suffices to minimize OP. By the triangle inequality, it is clear that OP is minimized when its extension passes through C. We can compute that $OC = 25$, so $OP \geq OC - PC = 25 - 6 = 19$, which means that $PQ+QR+RP \geq 2MN = \sqrt{2}OP \geq 19\sqrt{2}$. It is clear that such a setup is achievable, thus the answer is $\left(19\sqrt{2}\right)^2 = \boxed{722}$.



Proposed by Sebastian Zhu

10 Set 10 Solutions

28. The main idea of this problem is to subtract off the areas of triangles AEF, BDF, CED from triangle ABC. We have
$$\frac{[AEF]}{[ABC]} = \frac{[AEF]}{[AEB]} \cdot \frac{[AEB]}{[ABC]} = \frac{AF}{AB} \cdot \frac{AE}{AC},$$
where $[K]$ is the area of polygon K. Similarly,
$$\frac{[BDF]}{[ABC]} = \frac{BF}{BA} \cdot \frac{BD}{BC}$$
and
$$\frac{[CED]}{[ABC]} = \frac{CD}{CB} \cdot \frac{CE}{CA}.$$
Now we just need to compute a few side lengths. Since AE and AF are both tangents to the incircle, we must have $AE = AF$. Suppose $x = AE = AF$, $y = BF = BD$, and $z = CD = CE$. Then,
$$\begin{aligned} x + y &= AB = 7, \\ y + z &= BC = 12, \\ z + x &= CA = 13. \end{aligned}$$
Solving this system gives $x = 4, y = 3, z = 9$. Then,
$$\frac{[DEF]}{[ABC]} = 1 - \frac{[AEF]}{[ABC]} - \frac{[BDF]}{[ABC]} - \frac{[CED]}{[ABC]} = \frac{18}{91}.$$
This gives $18 + 91 = \boxed{109}$.



Proposed by William Yue

29. First, note that if Sebastian selects a token at (x, y, z), this token contributes a score of $S = \frac{1}{2^{x+y+z}}$ before replacing the token. After replacing the token, the three tokens together contribute a score of

$$3 \cdot \frac{1}{2^{x+y+z+1}} = \frac{3}{2}S.$$

Therefore it is most beneficial to perform moves on tokens with the highest score, which are the tokens on (x, y, z) with the lowest sum $x + y + z$. Let's say a token at position (x, y, z) has *value* $x + y + z$, so that replacing a token of value k produces three tokens of value $k + 1$. Then after the first move (forced), we have three tokens of value 1. Replacing these three tokens yields nine tokens of value 2, after four total replacements. Replacing these nine tokens yields 27 tokens of value 3, after 13 total replacements. Replacing these 27 tokens yields 81 tokens of value 4, after 40 total replacements. Now we have only 60 replacements left, so we can replace 60 of the 81 tokens of value 4, which yields 21 tokens of value 4 and 180 tokens of value 5. Finally, note that the score of a token given its value k is equal to $\frac{1}{2^k}$, so the final score of this set of tokens is

$$21 \cdot \frac{1}{2^4} + 180 \cdot \frac{1}{2^5} = \frac{111}{16}.$$

Then $111 + 16 = \boxed{127}$.

Proposed by William Yue

30. We approach this problem by recursion. Note that the digits reduce into $0, 1, 2, 3 \pmod 5$. We let a_n be the number of n digit numbers with the given digits, and with a digit sum equivalent to $0 \pmod 5$, and we define b_n, c_n, d_n, e_n analogously for $1, 2, 3, 4 \pmod 5$. Clearly, $a_1 = b_1 = c_1 = d_1 = 1$ and $e_1 = 0$. Then, for every new digit added,

$$a_n = c_{n-1} + d_{n-1} + e_{n-1} + a_{n-1},$$

as we can add a digit of any residue modulo 5 except 4. Now, we can set

$$s_n = a_n + b_n + c_n + d_n + e_n,$$

so

$$a_n = s_{n-1} - b_{n-1}.$$

Similarly,

$$b_n = s_{n-1} - c_{n-1},$$
$$c_n = s_{n-1} - d_{n-1},$$
$$d_n = s_{n-1} - e_{n-1},$$
$$e_n = s_{n-1} - a_{n-1}.$$

Now, we can plug these formulas into the right hand side of $a_n = s_{n-1} - b_{n-1}$, yielding

$$\begin{aligned}
a_n &= s_{n-1} - b_{n-1} \\
&= s_{n-1} - (s_{n-2} - c_{n-2}) \\
&= s_{n-1} - s_{n-2} + s_{n-3} - d_{n-3} \\
&= s_{n-1} - s_{n-2} + s_{n-3} - s_{n-4} + e_{n-4} \\
&= s_{n-1} - s_{n-2} + s_{n-3} - s_{n-4} + s_{n-5} - a_{n-5}
\end{aligned}$$

It suffices to compute s_n. However, this is just the total number of numbers with digits as $1, 5, 7$, or 8 (as the digit sum can be anything), so $s_n = 4^n$, as there are four choices for each digit. Thus,

$$a_6 = 4^5 - 4^4 + 4^3 - 4^2 + 4 - 1 = \frac{4^6 - 1}{5} = \boxed{819}.$$

Note. Here, we will present an elegant alternate method, utilizing complex numbers.

After using the same definitions of a_n, b_n, c_n, d_n, e_n as above, we let $\zeta = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = e^{\frac{2i\pi}{5}}$ be the first fifth root of unity. Then, notice that

$$a_n\zeta^0 + b_n\zeta^1 + c_n\zeta^2 + d_n\zeta^3 + e_n\zeta^4 = \sum_{x_1, x_2, \ldots, x_n \in \{1,5,7,8\}} \zeta^{x_1 + x_2 + \cdots + x_n}.$$

The idea here is that the right hand side takes in all inputs of n digit integers with digits $1, 5, 7, 8$, and the root of unity ζ takes the sum modulo 5, outputting the left hand side. Now, the right hand side is equal to

$$(\zeta^1 + \zeta^5 + \zeta^7 + \zeta^8)^n = (1 + \zeta + \zeta^2 + \zeta^3)^n,$$

and since $\sum_{i=0}^{4} \zeta^i = 0$, this is equal to $(-\zeta^4)^n$. If we take $n = 6$, we obtain

$$a_6\zeta^0 + b_6\zeta^1 + c_6\zeta^2 + d_6\zeta^3 + e_6\zeta^4 = \zeta^{24} = \zeta^4, \tag{1}$$

so

$$a_6\zeta^0 + b_6\zeta^1 + c_6\zeta^2 + d_6\zeta^3 + (e_6 - 1)\zeta^4 = 0$$

We proceed with a lemma.

Lemma. *For any prime p and integers $a_0, a_1, \cdots a_{p-1}$, if*

$$a_0\zeta^0 + a_1\zeta^1 + \cdots + a_{p-1}\zeta^{p-1} = 0,$$

then $a_0 = a_1 = \cdots = a_{p-1}$.

Proof. Just consider the polynomials $P(X) = a_0 + a_1X + a_2X^2 + \cdots + a_{p-1}X^{p-1}$ and $Q(X) = 1 + X + X^2 + \cdots + X^{p-1}$. It is pretty well known that $Q(X)$ is irreducible over $\mathbb{Q}$. In fact,

$$Q(X+1) = \frac{(X+1)^p - 1}{X} = \sum_{i=0}^{p-1} \binom{p}{p-i-1} X^i,$$

which is irreducible by Eisenstein's Criterion on the prime p, as $p \mid \binom{p}{i}$ for all $0 < i < p$. However, Q and P share the root ζ, so Q must divide P. Thus, $a_0 = a_1 = \cdots = a_{p-1}$. $\square$

We apply this lemma on (1), giving

$$a_6 = b_6 = c_6 = d_6 = e_6 - 1 = k$$

for some constant k. Then,

$$5k + 1 = a_6 + b_6 + c_6 + d_6 + e_6 = s_6 = 4^6,$$

so $a_6 = \dfrac{4^6 - 1}{5} = \boxed{819}$.

Proposed by William Yue

11 Set 11 Solutions

31. We break the sum into

$$\sum_{i=16}^{32} \frac{1}{T_i} + \sum_{i=16}^{32} \frac{1}{S_i} = 2\left(\sum_{i=16}^{32} \frac{1}{i(i+1)}\right) + \frac{3}{2}\left(\sum_{i=16}^{32} \frac{2}{i(i+2)}\right).$$

These fractions decompose into

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1} \text{ and } \frac{2}{i(i+2)} = \frac{1}{i} - \frac{1}{i+2}.$$

This allows us to telescope the sums into

$$2\left(\frac{1}{16} - \frac{1}{17} + \frac{1}{17} - \frac{1}{18} + \cdots + \frac{1}{32} - \frac{1}{33}\right) + \frac{1}{16} - \frac{1}{18} + \frac{1}{17} - \frac{1}{19} + \cdots + \frac{1}{32} - \frac{1}{34}$$
$$= 2\left(\frac{1}{16} - \frac{1}{33}\right) + \frac{3}{2}\left(\frac{1}{16} + \frac{1}{17} - \frac{1}{33} - \frac{1}{34}\right) = \frac{2815}{17952},$$

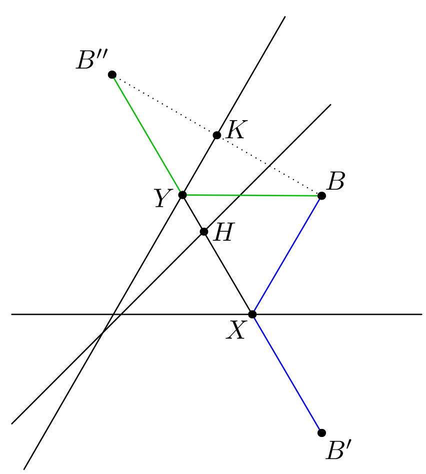
which results in $2815 + 17952 = \boxed{20767}$.

Proposed by William Yue

32. We reflect Will's barnhouse $B(24, 15)$ over the river at $y = 2$ to a point B', and over the strip mall at $y = \sqrt{3}x$ to a point B''. The idea now is that if Will's house is at H, he reaches the river at point X, and the strip mall at point Y, then the distance he travels is

$$HX + XB + BY + YH = HX + XB' + B''Y + YH.$$

This distance is clearly minimized when H lies on $B'B''$. Now it suffices to find the points B' and B'', and we can intersect lines $B'B''$ and $y = x$.



Clearly, $B' = (24, -11)$, since we reflect B over a horizontal line. As for B'', we will determine the foot of the perpendicular from B onto $y = \sqrt{3}x$, which we will call K. Since BK is perpendicular to $y = \sqrt{3}x$, its slope is $-\frac{1}{\sqrt{3}}$. Therefore, BK is defined by

$$y - 15 = -\frac{1}{\sqrt{3}}(x - 24).$$

Intersecting this line with $y = \sqrt{3}x$, we can solve the system to get

$$\begin{aligned}
\sqrt{3}x &= -\frac{\sqrt{3}}{3}x + 8\sqrt{3} + 15 \\
\frac{4\sqrt{3}}{3}x &= 15 + 8\sqrt{3} \\
x &= \frac{45 + 24\sqrt{3}}{4\sqrt{3}} = \frac{15\sqrt{3} + 24}{4}.
\end{aligned}$$

We can solve for y, so

$$K = \left(\frac{15\sqrt{3} + 24}{4}, \frac{45 + 24\sqrt{3}}{4}\right).$$

Since K is the midpoint of BB'', we can determine B'' to be

$$B'' = \left(\frac{15\sqrt{3} - 24}{2}, \frac{24\sqrt{3} + 15}{2}\right).$$

Now, we need to find the equation for $B'B''$. Its slope is

$$\frac{(24\sqrt{3} + 15)/2 - (-11)}{(15\sqrt{3} - 24)/2 - 24} = \frac{37 - 24\sqrt{3}}{-72 + 15\sqrt{3}},$$

so its defined by

$$y + 11 = \frac{37 + 24\sqrt{3}}{-72 + 15\sqrt{3}}(x - 24).$$

It remains to intersect this line with $y = x$. Solving,

$$\begin{aligned}
(-72 + 15\sqrt{3})x + 11(-72 + 15\sqrt{3}) &= (37 + 24\sqrt{3})x - 24(37 + 24\sqrt{3}). \\
(-109 - 9\sqrt{3})x &= -96 - 741\sqrt{3}.
\end{aligned}$$

Therefore

$$x = \frac{96 + 741\sqrt{3}}{109 + 9\sqrt{3}} = \frac{(96 + 741\sqrt{3})(109 - 9\sqrt{3})}{109^2 - (9\sqrt{3})^2} = \frac{79905\sqrt{3} - 9543}{11638},$$

which results in $79905 + 3 + 9543 + 11638 = \boxed{101089}$.

Proposed by William Yue

33. Take the equation $n^2 = (m+1)^3 - m^3$ for some integer m. Then

$$\begin{aligned}
n^2 &= 3m^2 + 3m + 1 \\
4n^2 - 1 &= 12m^2 + 12m + 3 \\
(2n+1)(2n-1) &= 3(2m+1)^2.
\end{aligned}$$

Because $2n + 1$ and $2n - 1$ are relatively prime, we know that one of these terms must be a perfect square and the other one must be three times a perfect square. However, we cannot have $2n - 1$ be three times a square, because then $2n + 1$ would be a perfect square congruent to $2 \pmod 3$, which is impossible. Therefore $2n - 1$ is a perfect square; say $2n - 1 = a^2$.

Now we proceed to the second equation. Set $2n + 287 = k^2$, and plug in $2n - 1 = a^2$ to get

$$\begin{aligned}
a^2 + 288 &= k^2 \\
288 &= (k + a)(k - a).
\end{aligned}$$

Since maximizing n is the same as maximizing a, we need only to maximize the difference between the two factors $k+a$ and $k-a$. We quickly see that 288 and 1 are impossible, as they do not produce integer values for k and a. However, 144 and 2 do work, yielding $k=73$ and $a=71$. Then

$$\begin{aligned} 2n-1 &= 71^2 \\ 2n-1 &= 5041 \\ n &= 2521, \end{aligned}$$

so the maximum possible value of n is $\boxed{2521}$. Plugging it back in shows that it indeed works, yielding $2521^2 = 1456^3 - 1455^3$ and $2 \cdot 2521 + 287 = 73^2$.

Proposed by Sebastian Zhu

12 Set 12 Solutions

34. https://en.wikipedia.org/wiki/Collatz_conjecture has some great information about the Collatz conjecture if you're interested. In case you were wondering, the Collatz sequence of $\boxed{670617279}$, the answer, has a length of 986 steps!

Proposed by Sebastian Zhu

35. We claim that

$$ex(n, K_i) = \frac{i-2}{i-1} \cdot \frac{n^2 - r^2}{2} + \binom{r}{2},$$

where r is the remainder when n is divided by $i-1$. In fact, this known as Turan's Theorem in Extremal Graph Theory. Note that equality can hold; take a complete multipartite graph of $i-1$ groups, r of which with $q+1$ elements, and $i-1-r$ of which with q elements, where $n = q(i-1)+r$. This is called a Turan Graph. Now, we will prove Turan's using the following lemma.

Lemma (Zarankiewicz's Lemma)**.** *If G is a graph of degree n without a subgraph K_i, then it contains a vertex with degree of at most $\left\lfloor \frac{i-2}{i-1} n \right\rfloor$.*

Proof. Let $N(v)$ denote the neighborhood of a vertex v; that is, the set of all vertices that are connected to v by an edge. For the sake of contradiction, assume otherwise. Consider an arbitrary vertex v_1. It must satisfy

$$|N(v_1)| > \left\lfloor \frac{i-2}{i-1} \cdot n \right\rfloor > 0,$$

so there exists $v_2 \in N(v_1)$. Then, points that are neighbors to both:

$$\begin{aligned} |N(v_1) \cap N(v_2)| &= |N(v_1)| + |N(v_2)| - |N(v_1) \cup N(v_2)| \\ &\geq 2\left(1 + \left\lfloor \frac{i-2}{i-1} \cdot n \right\rfloor\right) - n > 0, \end{aligned}$$

so there exists a $v_3 \in N(v_1) \cap N(v_2)$. We continue this for the rest of the vertices, and since

$$\bigcap_{j=1}^{k} N(v_j) \geq k\left(1 + \left\lfloor \frac{i-2}{i-1} \cdot n \right\rfloor\right) - (k-1)n > 0$$

for all k, we set $j = i-1$ so that we can choose a vertex

$$v_i \in \bigcap_{j=1}^{i-1} N(v_j),$$

but then $v_1, v_2, \ldots, v_i$ form a complete graph on i vertices, which violates the assumption. $\square$

Now we will use this lemma to prove Turan's Theorem. We induct on n. Suppose the statement is true for $n = k$. We will show it is true for $n = k+1$. Since the graph doesn't contain a K_i, applying Zarankiewicz's Lemma on the graph means we can choose a vertex x such that $\deg(x) \le \lfloor \frac{i-2}{i-1} \cdot (k+1) \rfloor$. If we remove this vertex and all edges attached to it, we get a graph of k vertices, and by our inductive hypothesis, the maximal number of edges in this graph is precisely

$$ex(k, K_i) = \frac{i-2}{i-1} \cdot \frac{k^2 - r_k^2}{2} + \binom{r_k}{2},$$

where r_k is the remainder when k is divided by $i-1$. Then, after adding back that vertex, it suffices to confirm that

$$\frac{i-2}{i-1} \cdot \frac{k^2 - r_k^2}{2} + \binom{r_k}{2} + \left\lfloor \frac{i-2}{i-1} \cdot (k+1) \right\rfloor = \frac{i-2}{i-1} \cdot \frac{(k+1)^2 - r^2}{2} + \binom{r}{2},$$

where r is the remainder when $k+1$ is divided by $i-1$, which is in fact true. This completes the inductive step. The Turan Graph suffices for every base case. Then, the problem reduces to computing

$$\sum_{i=2}^{2018} \frac{i-2}{i-1} \cdot \frac{2018^2 - (r(2018, i-1))^2}{2} + \binom{r(2018, i-1)}{2} = \boxed{4090111833},$$

where $r(2018, i-1)$ is the remainder when 2018 is divided by $i-1$.

Proposed by William Yue

36. No solution for this problem. We are unaware of any strategy that would net you the greatest expected number of points, but if you find one, we'd be happy to hear it!

Proposed by William Yue and Sebastian Zhu