MOAA 2018 Individual Round Solutions

Andover Math Club

1. This can be factored as $(20+1)(18+1) = 21 \cdot 19 = (20-1)(20+1) = 20^2 - 1 = \boxed{399}$.

   Proposed by William Yue

2. Since the choices of ice cream flavor, cone type, and toppings are independent, the total number of ice cream combinations is equal to product of the number of individual choices, so there are a total of $10 \cdot 5 \cdot 5 = \boxed{250}$ possibilities.

   Proposed by Justin Chang

3. $(a+b)^2 - (a-b)^2 = a^2 + 2ab + b^2 - (a^2 - 2ab + b^2) = 4ab$. Therefore, the expression is equivalent to $\frac{4a^2b^2}{4ab} = ab = 7 \cdot 77 = \boxed{539}$.

4. Remember that adding x percent to a quantity is the same as multiplying the quantity by $1 + \frac{x}{100}$. The answer is $100000 \cdot 0.8 \cdot 1.25 \cdot 0.75 \cdot 1.6 = \boxed{120000}$.

   Proposed by Justin Chang

5. We can simply calculate the area of $ABCD$ and subtract out the areas of $\triangle ANK$, $\triangle BKL$, $\triangle CLM$, and $\triangle DMN$ to get the area of $KLMN$.

   - The area of $ABCD$ is 25.
   - The area of $\triangle ANK$ is $1/2 \cdot 1 \cdot 1 = 1/2$.
   - The area of $\triangle BKL$ is $1/2 \cdot 4 \cdot 3 = 6$.
   - The area of $\triangle CLM$ is $1/2 \cdot 2 \cdot 2 = 2$.
   - The area of $\triangle DMN$ is $1/2 \cdot 3 \cdot 4 = 6$.

   Finally, the area of $KLMN$ is
   $$25 - \frac{1}{2} - 6 - 2 - 6 = \frac{21}{2},$$
   giving an answer of $21 + 2 = \boxed{23}$.

   Proposed by Vincent Fan

6. The list of prime numbers less than 30 is $2, 3, 5, 7, 11, 13, 17, 19, 23, 29$. The only candidates that sum to 30 when paired together are $7+23 = 11+19 = 13+17 = 30$. The answer is $7 \cdot 23 + 11 \cdot 19 + 13 \cdot 17 = \boxed{591}$.

   Note. Many people gave the answer 1182 to this problem, probably arising from the fact that they independently added $7 \cdot 23$ and $23 \cdot 7$, etc. Technically this answer is incorrect, since the problem asked for all possible values of pq independent of the order of p and q; nevertheless, we have decided to accept $\boxed{1182}$ as a correct answer as well.

   Proposed by Justin Chang

7. Let the vertices of Tori's triangle be A, B, C such that $AB = 15, BC = 25, CA = 20$. Then let the foot of the altitude from A be H. By AA Similarity, $\triangle ABC \sim \triangle HBA$. Therefore, $\frac{AB}{BC} = \frac{BH}{AB}$. Substituting the values of AB and BC yields $BH = 9$, and thus $HC = BC - BH = 16$. Since the area of $\triangle ABC$ can be written as $[ABC] = AB \cdot AC/2 = BC \cdot AH/2$, we find that $AH = 12$. Therefore, the difference in areas between the triangles HCA and HBA is $\frac{1}{2} \cdot 12 \cdot 16 - \frac{1}{2} \cdot 12 \cdot 9 = \frac{1}{2} \cdot 12 \cdot 7 = \boxed{42}$.

Proposed by Andy Xu

8. Any number that ends in three zeroes must be divisible by 1000 must be divisible by 125 but not by 625; we can ignore powers of two since any factorial greater than or equal to $4! = 24$ is divisible by 8. Therefore, a satisfactory factorial $k!$ must have $15 \leq k \leq 19$. The answer is $19 + 15 = \boxed{34}$.

Proposed by Sam Sheehan

9. Note that Sam needs to traverse 8 horizontal segments and 8 vertical segments to minimize the distance of the path. We can use complementary counting to account for the omitted point in the center of the grid. There are $\binom{16}{8}$ ways to choose a path if Sam is allowed to walk anywhere and $\binom{8}{4} \cdot \binom{8}{4}$ ways for Vincent to choose a path that has to go through the center of the grid. The answer is $\binom{16}{8} - \binom{8}{4} \cdot \binom{8}{4} = \boxed{7970}$.

   Remark. We define the binomial coefficient as $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, which is equal to the number of ways to select k items from n items.

Proposed by Vincent Fan

10. Knowing that Mr. Iyer draws the only red marble and one of the two yellow marbles, there are four marbles unaccounted for: one yellow marble, and three blue marbles. The probability that Mr. Iyer lost two blue marbles and drew one yellow and then one red marble is equal to $(\frac{3}{6} \cdot \frac{2}{5})(\frac{2}{4} \cdot \frac{1}{3}) = \frac{1}{30}$, whereas the probability that Mr. Iyer lost one blue and one yellow marble and then drew one yellow and then one yellow marble is equal to $(\frac{3}{6} \cdot \frac{2}{5} \cdot 2)(\frac{1}{4} \cdot \frac{1}{3}) = \frac{1}{30}$. Thus, we see that both cases are equally likely. In the first case, the probability that the next marble is blue is $\frac{1}{2}$, whereas in the second case, the probability that the next marble is blue is 1. Thus, the answer is $\frac{1+1/2}{2} = \frac{3}{4}$, so the answer is $3 + 4 = \boxed{7}$.

Proposed by Wendy Wu

11. Let $a = 1$. Then the expression equals $(1 + 1)(1 + 2)(1 + 3) = 24$. Let $a = 2$. Then the expression equals $(8 + 1)(8 + 2)(8 + 3) = 990$. $\gcd(24, 990) = 6$, so the answer must be a factor of 6. Let the positive integer $b = a^3$. We note that, of $b(b + 1)(b + 2)$, at least one of the three factors must be divisible by 3 and at least one of the factors must be divisible by 2. Therefore, the product must be divisible by 6 and divide into 6, so the answer is $\boxed{6}$.

Proposed by Justin Chang

12. Note that 160401 can be written as $20^4 + 20^2 + 1$. Since $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$, we can further factor the expression:

$$160401 = (20^2 + 20 + 1)(20^2 - 20 + 1) = 421 \cdot 381.$$

421 is prime so the answer is $\boxed{421}$.

Alternatively, note that

$$160401 = 401^2 - 20^2,$$

so that $160401 = (401 + 20)(401 - 20) = (421)(381)$. Thus the answer is 421.

Proposed by Jeffrey Shi

13. The discriminant of this quadratic must be less than or equal to 0 for there to be no solutions. Therefore, $m^2 - 4 \cdot 2018 \leq 0$ and $m^2 \leq 8072$. Since m must be an integer, we see that $m^2 < 8100$ and $-90 < m < 90$. The answer is $\boxed{179}$.

Proposed by Andy Xu

14. Note that the largest possible product of two two-digit numbers is $99 \cdot 99 = 9801$. Note also that any four-digit palindrome can be written in the form $abba$, and is thus a multiple of 11. Note also that $99 \cdot 91 = 9009$, so if there existed a larger such palindrome then both two-digit numbers would have to be greater than 90 and at least one would be a multiple of 11. The only multiple of 11 greater than 90 is 99, so any four-digit palindrome greater than 9009 must be divisible by 99 as well. The only such palindrome is $9999 > 9801$, so the largest possible palindrome that can be written as the product of two two-digit palindromes is $\boxed{9009}$.

Proposed by Justin Chang

15. By power of a point, we see that

$$\begin{aligned} EA \cdot EB &= EC \cdot ED \\ \Rightarrow EA &= \frac{EC \cdot ED}{EB} = \frac{3 \cdot 4}{2} = 6. \end{aligned}$$

Therefore $AB = 8$. Now let O be the center of ω. Let the feet of the perpendiculars from O onto AB and CD be F and G, respectively, which are clearly the midpoints of their respective sides. Therefore,

$$OF = GE = ED - GD = 4 - \frac{7}{2} = \frac{1}{2}.$$

Then, by the Pythagorean Theorem,

$$r^2 = OA^2 = FA^2 + FO^2 = \frac{65}{4},$$

so our answer is $65 + 4 = \boxed{69}$.

Proposed by William Yue

16. Note that the given equality can be rearranged to

$$(a - 2b)^2 + (b - 3c)^2 + (6c - a)^2 = 0,$$

from which it follows that $a = 2b = 6c$ since a, b, c are real. It is possible for all three of these equations to be satisfied (take $(a, b, c) = (6, 3, 1)$ for example). Therefore, we can determine the answer:

$$\begin{aligned} \frac{9(a+b+c)^3}{5abc} &= \frac{9(6c+3c+c)^3}{5 \cdot 6c \cdot 3c \cdot c} \\ &= \frac{9000c^3}{90c^3} \\ &= \boxed{100}. \end{aligned}$$

Proposed by Sebastian Zhu

17. We note that since the parity of $|x|$ is the same as that of x, n must be even. We show that all even n satisfy the problem condition as follows. Let $n = 2m$, and let $4i - 2 = k_{2i-1} = -k_{2i}$ for $1 \leq i \leq m$. Then $k_1 + k_2 + \cdots + k_{n-1} + k_n$ certainly equals 0, and

$$|k_1| + |k_2| + \cdots + |k_{n-1}| + |k_n| = 2((4-2) + (8-2) + (12-2) + \cdots + (4m-2))$$

$$= 4(1 + 3 + 5 + \cdots + (2m-1)) = 4m^2 = n^2,$$

where we use the fact that the sum of the first m odd integers equals m^2. The construction is finished upon noting that all k_i are distinct. There are $\boxed{499999}$ positive even integers less than 10^6.

Proposed by Sebastian Zhu

18. We see that $|(x^2-4x)-(y^2-4y)| = |(x^2-4x)+(y^2-4y)|$, which means that either $x^2 - 4x = 0$ or $y^2 - 4y = 0$. In the case that $x^2 - 4x = 0$, $x = 0, 4$ and $|y^2 - 4y| = 4$. Thus, $y^2 - 4y - 4 = 0$ (where $y = \frac{4 \pm \sqrt{32}}{2}$) or $y^2 - 4y + 4 = 0$ (where $y = 2$). The sum of all x and y under this case is equal to $3(0+4)+2(\frac{4+\sqrt{32}}{2}+\frac{4-\sqrt{32}}{2}+2) = 24$. Note that owing to the absolute values, the system of equations is symmetric around $x = y$, so we know the sum of all x and y in the case that $y^2 - 4y = 0$ is also 24. The final answer is the sum of the values in these two cases, which is $24 + 24 = \boxed{48}$

Proposed by William Duan

19. Let a_n be the number of such sequences of length n ending in W, b_n be the number of such sequences of length n ending in Q, and c_n be the number of such sequences of length n ending in N. We derive recursions for each of these sequences.
Clearly

$$a_n = b_{n-1} + c_{n-1},$$

because any sequence ending in W must have a letter in the second to last position not equal to W.
Also,

$$b_n = a_{n-1} + c_{n-1} + a_{n-2} + c_{n-2}.$$

This is because if a sequence ends in Q, either its second to last letter is a Q or it is not. Assuming it is not, this becomes the first case and we simply add $a_{n-1} + c_{n-1}$. Assuming it is, then the letter before that one must not be Q, so there are $a_{n-2} + c_{n-2}$ cases there.
Finally,

$$c_n = b_{n-1} + a_{n-1} + b_{n-2} + a_{n-2} + b_{n-3} + a_{n-3}.$$

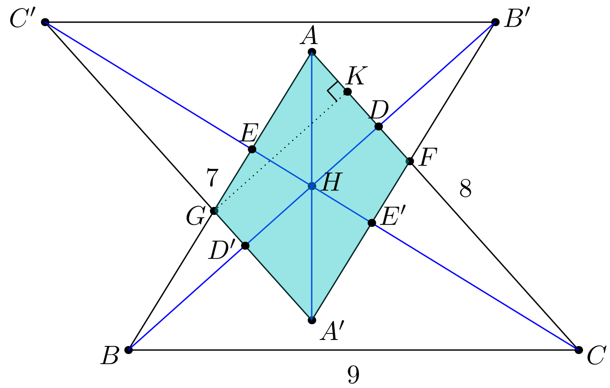
The reasons are similar to those stated above, with the first two terms corresponding to sequences ending in exactly one N, the second two terms corresponding to sequences ending in exactly two N's, and the last two terms corresponding to sequences ending in exactly three N's.
Now we construct a chart and plug in some initial conditions to find:

n	1	2	3	4	5	6	7	8
a_n	1	2	6	15	39	101	261	676
b_n	1	3	7	19	49	126	327	846
c_n	1	3	8	20	52	135	349	903
Total	3	8	21	54	130	362	937	2425

Therefore the answer is $\boxed{2425}$.

Proposed by Vincent Fan

20. Let D and E be the feet of the altitudes from B and C, respectively. Also, let F be the intersection of $A'B'$ with AC and let G be the intersection of $A'C'$ with AB. Since the diagonals of quadrilateral $BCB'C'$ bisect each other, we must have $BCB'C'$ be a parallelogram, so $BC \parallel B'C'$, and similarly, $AB \parallel A'B'$ and $CA \parallel C'A'$, and since $BC = B'C'$, $\triangle ABC \cong \triangle A'B'C'$.



Now we have to compute quite a few lengths, and the goal is to determine HE and HD. By Heron's Formula, the area of ABC is

$$[ABC] = \sqrt{12(12-7)(12-8)(12-9)} = 12\sqrt{5}.$$

Then, $BD = \frac{2[ABC]}{AC} = 3\sqrt{5}$ and $CE = \frac{2[ABC]}{AB} = \frac{24\sqrt{5}}{7}$. By the Pythagorean Theorem,

$$AD = \sqrt{AB^2 - BD^2} = \sqrt{7^2 - (3\sqrt{5})^2} = 2,$$

and

$$AE = \sqrt{AC^2 - CE^2} = \sqrt{8^2 - \left(\frac{24\sqrt{5}}{7}\right)^2} = \frac{16}{7}.$$

Then, $BE = 7 - AE = \frac{33}{7}$ and $CD = 8 - AD = 6$. Now, since $\triangle BEH \sim \triangle BDA$ and $\triangle CDH \sim \triangle CEA$, we must have

$$\frac{HE}{BE} = \frac{AD}{BD} \Rightarrow HE = \frac{AD \cdot BE}{BD} = \frac{66/7}{3\sqrt{5}} = \frac{22\sqrt{5}}{35}$$

and

$$\frac{DH}{CD} = \frac{AE}{CE} \Rightarrow DH = \frac{AE \cdot CD}{CE} = \frac{96/7}{24\sqrt{5}/7} = \frac{4\sqrt{5}}{5}.$$

Let D' be the reflection of D over H. Since D lies on AC, we must have, by symmetry, D' lies on $A'C'$. Since $\angle HDA = \angle H'D'A' = 90^\circ$, $D'D$ is the distance between parallel lines AC and $A'C'$. Therefore, if K is the foot of the altitude from G onto AC,

$$GK = DD' = 2HD = \frac{8\sqrt{5}}{5}.$$

Now we just need to compute AG. However, $\triangle AGK \sim \triangle ABD$, so

$$\frac{AG}{GK} = \frac{AB}{BD} \Rightarrow AG = \frac{AB \cdot GK}{BD} = \frac{56}{15}.$$

Also, $EE' = 2HE$ is the distance between parallel lines AB and $A'B'$, so the area of $AGA'F$ is

$$[AGA'F] = EE' \cdot AG = \frac{352\sqrt{5}}{75}.$$

Therefore, our answer is $352 + 5 + 75 = \boxed{432}$.

Proposed by William Yue