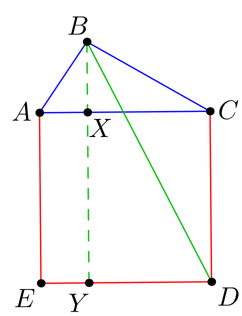
MOAA 2018 Team Round Solutions

Andover Math Club

1. Let the feet of the perpendiculars from B to AC and DE be X and Y, respectively. By Heron's Formula,
$$[ABC] = \sqrt{7 \cdot 1 \cdot 2 \cdot 4} = 2\sqrt{14}.$$
Therefore, $BX = \frac{2\sqrt{14}}{3}$. By the Pythagorean Theorem,
$$CX = \sqrt{25 - \frac{56}{9}} = \frac{13}{3}.$$
To finish, notice that $YD = XC$ and $BY = 6 + BX$, so
$$BD^2 = BY^2 + YD^2 = \left(\frac{2\sqrt{14}+18}{3}\right)^2 + \left(\frac{13}{3}\right)^2 = 61 + 8\sqrt{14}.$$
Our answer is $61 + 8 + 14 = \boxed{83}$.



Proposed by William Yue

2. Note that
$$16 = x^2 + 2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2$$
$$4 = x + \frac{1}{x},$$
since $x > 0$. In order to construct higher powers of x and $\frac{1}{x}$, we perform the following manipulation:
$$\left(x^2 + \frac{1}{x^2}\right)\left(x + \frac{1}{x}\right) = x^3 + \frac{1}{x^3} + x + \frac{1}{x}$$
$$x^3 + \frac{1}{x^3} = 14 \cdot 4 - 4$$
$$= 52.$$

We repeat the process:

$$\left(x^3+\frac{1}{x^3}\right)\left(x+\frac{1}{x}\right)=x^4+\frac{1}{x^4}+x^2+\frac{1}{x^2}$$
$$x^4+\frac{1}{x^4}=52\cdot 4-14$$
$$=194.$$

Finally:

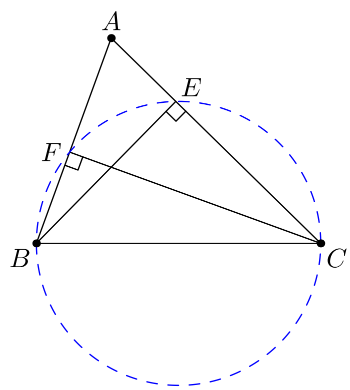
$$\left(x^4+\frac{1}{x^4}\right)\left(x+\frac{1}{x}\right)=x^5+\frac{1}{x^5}+x^3+\frac{1}{x^3}$$
$$x^5+\frac{1}{x^5}=194\cdot 4-52$$
$$=\boxed{724}.$$

Proposed by Shen Duan

3. We know that $\angle BFC = 90^\circ = \angle BEC$, because BE and CF are altitudes. This means that $BFEC$ is cyclic (can be inscribed in a circle) with its center on the midpoint of BC. Therefore we can apply power of a point on A with respect to the circumcircle of $BFEC$. This implies that

$$AF(AF+31)=84\cdot 24=2016.$$

Thus we can conclude that $AF=\boxed{32}$.



Proposed by William Yue

4. Let's examine some easy cases first. If Michael chooses 99, then Andrew automatically loses. Thus 99 is a losing position, which means that if it's your turn to move and 99 is written on the board, then, no matter what you do, you automatically lose under best play (in this case your opponent doesn't even get a chance to mess up). Noting this, when Michael picks any number between 91 and 98 inclusive, Andrew can just increment this integer to 99, thus handing Michael a losing position. Therefore all integers between 91 and 98 are winning positions, which means that you have some strategy that guarantees you to win no matter what your opponent does.

Let's take a step back and consider exactly what Michael is trying to achieve. Michael wants to reach a winning position, which is equivalent to handing Andrew a losing position. Therefore if Michael wants to win, he must choose a losing position to begin, since then Andrew will lose. Examining what we have discovered so far, we see that if Michael chooses 90, then Andrew is forced to hand a winning position to Michael. Therefore 90 is also losing, which forces $82-89$ to be winning. Continuing with this reasoning, we see that all multiples of 9 are losing, and everything else is winning. This makes sense, because if you are handed something other than a multiple of 9, you can always choose an integer between 1 and 8 to make the number a multiple of 9. On the other hand, if you receive a multiple of 9, then you will have no choice but to hand the other person a number which is not a multiple of 9. Finally, we note that the game must eventually terminate, as every move increments the integer by at least 1. Therefore our answer is the sum of all multiples of 9 less than 100, which is

$$9 + 18 + \cdots + 99 = 9 \cdot 66 = \boxed{594}.$$

Proposed by Sebastian Zhu

5. The problem is nowhere near as hard as it seems. The expected number of problems Mr. DoBa solves in a minute while listening to music is $0.6 \cdot \frac{1}{3} + 0.4 \cdot \frac{1}{5} = 0.28$. But the average number of problems he solves in a minute when not listening to music must also equal this number! Thus $m = 0.28$, and so $1000m = \boxed{280}$.

Proposed by Sebastian Zhu

6. We will assume, without loss of generality, that $m \geq n$ and the grid is m rows of n. Let's first count R. A rectangle in this grid is uniquely defined by two horizontal grid lines and two vertical grid lines. The number of ways to choose two horizontal grid lines is $\binom{m+1}{2}$, and the number of ways to choose two vertical grid lines is $\binom{n+1}{2}$, so

$$R = \binom{n+1}{2} \cdot \binom{m+1}{2} = \frac{mn(m+1)(n+1)}{4}.$$

Now we will count S. We will consider cases of what the side length of the square is. Suppose it is i, with $1 \leq i \leq n$ (as $n \leq m$). There are $n - i + 1$ ways to choose its vertical position, and $m - i + 1$ ways to choose its horizontal position. Thus, there are $(n - i + 1)(m - i + 1)$ squares of side length i in the grid. Summing over all i from 1 to n gives

$$S = \sum_{i=1}^{n} (n - i + 1)(m - i + 1) = \sum_{j=1}^{n} j(m - n + j),$$

where $j = n - i + 1$. Note that summing from $i_1 = 1$ to $i_2 = n$ is equivalent to summing from $j_2 = n - i_1 + 1 = n$ to $j_1 = n - i_2 + 1 = 1$. We now break this summation into two:

$$\begin{aligned}
\sum_{j=1}^{n} j(m - n + j) &= \sum_{j=1}^{n} (m - n)j + \sum_{j=1}^{n} j^2 \\
&= (m - n) \sum_{j=1}^{n} j + \sum_{j=1}^{n} j^2 \\
&= (m - n)\frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6} \\
&= \frac{n(n+1)}{6}(3m - n + 1).
\end{aligned}$$

Therefore,

$$\frac{759}{50} = \frac{\frac{n(n+1)m(m+1)}{4}}{\frac{n(n+1)}{6}(3m - n + 1)} = \frac{3m(m+1)}{2(3m - n + 1)}.$$

Thus,

$$253(3m - n + 1) = 25m(m + 1).$$

Expanding and combining like terms gives

$$25m^2 + 25m = 759m - 253n + 253$$
$$25m^2 - 734m = 253 - 253n.$$

Since the right hand side is at most 0, the left hand side is too. Thus,

$$25m^2 - 734m \le 0 \Rightarrow 25m \le 734,$$

so $m \le 29$. Also, since $25m(m+1) = 253(3m-n+1)$, we must have $253 = 11 \cdot 23$ divide into $m(m+1)$. Clearly the only $m \le 29$ that satisfies this is $m = 22$. Plugging this back in gives

$$25 \cdot 22 \cdot 23 = 11 \cdot 23(67 - n),$$

so $67 - n = 50$, which implies $n = 17$. Therefore, $mn = 22 \cdot 17 = \boxed{374}$.

Proposed by William Yue

7. Consider each integer as a base 13 integer. Then the operation is equivalent to "chopping off" the last digit of this number at every step in its 13-pop. Therefore the condition reduces to finding the number of base 13 integers with at most 2018 digits with no 0s in their base 13 representation; this is because if there is a 0 in the base 13 representation of the integer, then eventually it will become the units digit of some number in its 13-pop, and consequently be divisible by 13. This total number of integers can easily be found to be

$$12 + 12^2 + 12^3 + \cdots + 12^{2018} = \frac{12(12^{2018} - 1)}{11}$$

by casework on the number of digits. It remains to calculate this number $\pmod{1000}$. It is clear that the number is congruent to $4 \pmod 8$, so we will focus on the number $\pmod{125}$. Observe the following:

$$\begin{aligned}
\frac{12(12^{2018}-1)}{11} &\equiv 12 \cdot 91(12^{2018} - 1) &\pmod{125}\\
&\equiv 92(12^{18} - 1) &\pmod{125}\\
&\equiv 92(19^9 - 1) &\pmod{125}\\
&\equiv 92(14^4 \cdot 19 - 1) &\pmod{125}\\
&\equiv 92(71^2 \cdot 19 - 1) &\pmod{125}\\
&\equiv 92(41 \cdot 19 - 1) &\pmod{125}\\
&\equiv 92 \cdot 28 &\pmod{125}\\
&\equiv 76 &\pmod{125}.
\end{aligned}$$

Since $76 \equiv 4 \pmod 8$, we see that $\dfrac{12(12^{2018}-1)}{11} \equiv 76 \pmod{1000}$, so our answer is $\boxed{76}$.

Proposed by Sebastian Zhu

8. We can simplify the first radical as follows:

$$\begin{aligned}
\sqrt{1 + \frac{\sqrt{3}}{2}} &= \sqrt{\frac{4 + 2\sqrt{3}}{4}}\\
&= \frac{\sqrt{4 + 2\sqrt{3}}}{2}\\
&= \frac{\sqrt{3} + 1}{2}.
\end{aligned}$$

Similarly,

$$\sqrt{1-\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}-1}{2}.$$

Thus, we can rewrite the problem as:

$$\frac{k}{2} = \left(\frac{\sqrt{3}+1}{2}\right)^x + \left(\frac{\sqrt{3}-1}{2}\right)^x$$

Now note that $(\frac{\sqrt{3}+1}{2}) + (\frac{\sqrt{3}-1}{2}) = \sqrt{3}$ and $(\frac{\sqrt{3}+1}{2}) \cdot (\frac{\sqrt{3}-1}{2}) = \frac{1}{2}$. Using this, we can write:

$$\begin{aligned}\left(\left(\frac{\sqrt{3}+1}{2}\right)^n + \left(\frac{\sqrt{3}-1}{2}\right)^n\right)\cdot\sqrt{3} &= \left(\left(\frac{\sqrt{3}+1}{2}\right)^n + \left(\frac{\sqrt{3}-1}{2}\right)^n\right)\cdot\left(\frac{\sqrt{3}+1}{2}+\frac{\sqrt{3}-1}{2}\right)\\ &= \left(\frac{\sqrt{3}+1}{2}\right)^{n+1} + \left(\frac{\sqrt{3}-1}{2}\right)^{n+1} + \frac{1}{2}\left(\left(\frac{\sqrt{3}+1}{2}\right)^{n-1} + \left(\frac{\sqrt{3}-1}{2}\right)^{n-1}\right)\end{aligned}$$

If we let $a_n = (\frac{\sqrt{3}+1}{2})^n + (\frac{\sqrt{3}-1}{2})^n$, we can write:

$$a_n\sqrt{3} = a_{n+1} + \frac{1}{2}a_{n-1}$$

Using this recursion, we can calculate that:

$$\begin{aligned}&a_1 = \sqrt{3},\ a_2 = 2,\ a_3 = \frac{3\sqrt{3}}{2},\\ &a_4 = \frac{7}{2},\ a_5 = \frac{11\sqrt{3}}{4},\ a_6 = \frac{13}{2}.\end{aligned}$$

The solutions we seek are the ones that have at most one factor of 2 in the denominator. Now we will prove that for all k, the power of 2 in the denominator of a_{k+4} will be 2 greater than the power of 2 in the denominator of a_k. This way, once we have a term, a_n, with exactly one power of 2 in the denominator, we know that the term a_{n+4} and all subsequent terms a_{n+4m} will have powers of 2 greater than one in their denominators.
To prove this, we will solve the recursion for a_{k+4} in terms of a_k and a_{k+1}.
We can write:

$$\begin{aligned}a_{k+2} &= a_{k+1}\sqrt{3} - \frac{1}{2}a_k,\\ a_{k+3} &= a_{k+2}\sqrt{3} - \frac{1}{2}a_{k+1}\\ &= (a_{k+1}\sqrt{3} - \frac{1}{2}a_k)\sqrt{3} - \frac{1}{2}a_{k+1}\\ &= \frac{5}{2}a_{k+1} - \frac{\sqrt{3}}{2}a_k,\\ a_{k+4} &= a_{k+3}\sqrt{3} - \frac{1}{2}a_{k+2}\\ &= (a_{k+3}\sqrt{3} - \frac{1}{2}a_{k+2})\sqrt{3} - \frac{1}{2}(a_{k+1}\sqrt{3} - \frac{1}{2}a_k)\\ &= 2a_{k+1}\sqrt{3} - \frac{5}{4}a_k.\end{aligned}$$

Now, let $a_k = \frac{b_k}{2^m}$ and $a_{k+1} = \frac{b_{k+1}}{2^n}$. Plugging these values into our new recursion gives:

$$a_{k+4} = \frac{b_k\sqrt{3}}{2^{m-1}} + \frac{5b_{k+1}}{2^{n+2}}$$

We can see that $n+2 > m-1$ because the exponent of 2 in the denominator can never incerase by 3. Using this, we can place the two fraction under a common denominator of 2^{n+2} by multiplying the first term by $2^{n+2-(m-1)} = 2^{n-m+3}$, giving us:

$$a_{k+4} = \frac{2^{n-m+3}b_{k+1}\sqrt{3} - 5b_k}{2_{n+2}}$$

The numerator must be odd since and even number minus an odd number is odd, so the power of 2 in the denominator of a_{k+4} is 2 greater than that of a_k, which concludes our proof. Since the terms a_3 through a_6 all have a denominator of at least 2, the only possible integral values of k result from a_2, a_4, and a_6, giving a sum of $4 + 7 + 13 = \boxed{24}$

Proposed by Max Tao

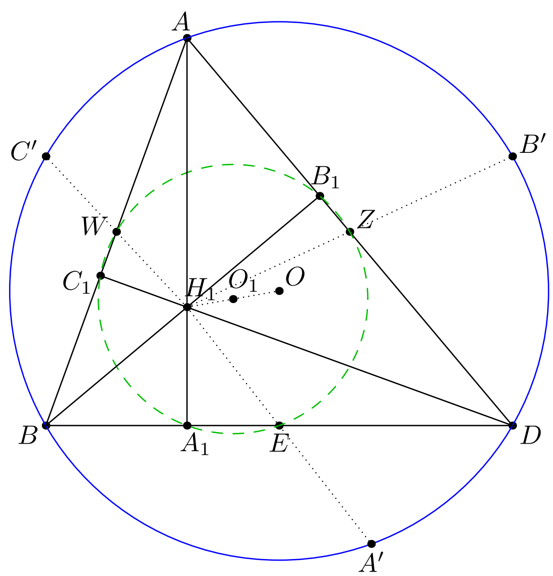
9. Let H_1 and H_2 be the respective orthocenters of ABD and CBD. We begin by proving the following claim:

Lemma. *O_1 is the midpoint of the segment H_1O, where O is the circumcenter of ABD.*

Proof. We will show the reflections of H_1 over E, Z, W lie on the circumcircle of ABC. In fact, if A' is the reflection of H_1 over E, then

$$\angle BA'D = \angle BH_1D = \angle C_1H_1B_1 = 180^\circ - \angle BAD,$$

so $ABA'D$ is cyclic. Thus, there exists a homothety centered at H_1 of scale factor two that sends the circumcircle of EZW to the circumcircle of ABC. This also sends the center of the first circle, O_1 to the center of the second circle, O, so $H_1O = 2H_1O_1$ and O_1 lies on OH_1. Therefore, O_1 is the midpoint of OH_1. □



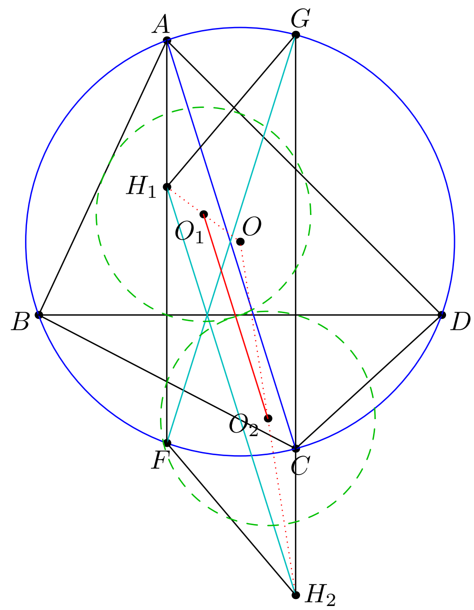
Similarly, O_2 is the midpoint of OH_2. Therefore, $O_1O_1 = \frac{1}{2}H_1H_2$. Now, let F and G be the reflections of H_1 and H_2 over BD, respectively. Since

$$\angle BFD = \angle BH_1D = 180^\circ - \angle BAD,$$

F lies on the circumcircle of $ABCD$. Similarly, G does too. However, $AF \parallel CG$, as they are both perpendicular to BD. Therefore,

$$H_1H_2 = FG = AC = 800,$$

where the first equality is true because of reflection, and the second is true since $AFCG$ is a cyclic (and therefore isosceles) trapezoid. Thus, $O_1O_2 = \frac{1}{2} \cdot 800 = \boxed{400}$.



Note. In this problem, O_1 is known as the *nine-point center* of ABD. This is because its the center of the *nine-point circle*, which passes through following nine points in a triangle ABC with orthocenter H:

- The midpoints of the sides BC, CA, AB
- The feet of the altitudes from A, B, C
- The midpoints of AH, BH, CH

A proof of the existence of the nine-point circle is very similar to the proof of our lemma; it uses a homothety centered at H that sends nine points on the circumcircle of ABC to the nine desired points of the nine-point circle.

Proposed by William Yue

10. Suppose that k satisfies the problem statement. In order for Vincent to cross the threshold $\frac{k}{2016}$ in ratio of green balls to total balls, he must choose a green ball in the position that allows him to pass that threshold. Therefore it must be true that sometime during his game, the bag contains m green balls out of n total balls such that

$$\frac{m}{n} < \frac{k}{2016}$$

and

$$\frac{m+1}{n+1} > \frac{k}{2016}.$$

The first inequality becomes $2016m < nk$, and the second inequality becomes $2016m > nk + k - 2016$. However, since m and n are integers, we can strengthen each of these inequalities to become

$$2016m \leq nk - 1$$

and

$$2016m \geq nk + k - 2015.$$

Thus

$$nk - 1 \geq 2016m \geq nk + k - 2015$$
$$k \leq 2014.$$

It seems like this is the entire solution set for the problem. However, this approach is actually the wrong one! Look a little more closely at the inequality

$$nk - 1 \geq 2016m \geq nk + k - 2015$$
$$2015 - k \geq nk - 2016m \geq 1.$$

Using Bézout's Lemma, the smallest attainable positive value of $nk - 2016m$ is $\gcd(k, 2016)$ as n and m vary. Therefore, k satisfies the problem statement if and only if $\gcd(k, 2016) \leq 2015 - k$. Note that

$$\gcd(k, 2016) = \gcd(2016, 2016 - k).$$

This follows from the Euclidean algorithm. Therefore our goal is to count the k which satisfy

$$\gcd(2016, 2016 - k) \leq (2016 - k) - 1$$
$$\gcd(2016, 2016 - k) < 2016 - k.$$

Equivalently, let us count the k that do not satisfy this inequality. Note that

$$\gcd(2016, 2016 - k) \leq 2016 - k,$$

with equality if and only if $2016 - k$ divides 2016. Therefore the values of k that do not satisfy the inequality are precisely those in which $2016 - k$ divides 2016. To finish, note that if $2016 - k = d$, where d is a divisor of 2016, then $k = 2016 - d$. Thus our answer is

$$\begin{aligned}
&1 + 2 + \cdots + 2015 - (2016 - 1) - (2016 - 2) - \cdots - (2016 - 1008) - (2016 - 2016) \\
&= 1008 \cdot 2015 - 2016 \cdot d(2016) + \sigma(2016) \\
&= 1008 \cdot 2015 - 2016 \cdot 36 + 63 \cdot 13 \cdot 8 \\
&= 504(4030 - 144 + 13) \\
&= 504 \cdot 3899 \\
&= 2^3 \cdot 3^2 \cdot 7^2 \cdot 557,
\end{aligned}$$

using the fact that the number of divisors $d(2016)$ of $2016 = 2^5 \cdot 3^2 \cdot 7$ is $6 \cdot 3 \cdot 2 = 36$ and the sum of the divisors $\sigma(2016)$ of 2016 is $(1 + 2 + 4 + 8 + 16 + 32)(1 + 3 + 9)(1 + 7) = 63 \cdot 13 \cdot 8 = 6552$. Since 557 is prime, the answer is $\boxed{557}$.

Proposed by Sebastian Zhu