MOAA 2019 Sample Problems

MOAA Problem Writers

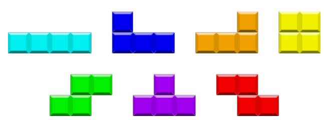
July 3, 2019

1 Problems

1. Find the value of the series
$$3 + 6 + 9 + 12 + \cdots + 2019.$$

2. Sam wants to eat a total of 43 lamb skewers, and he wants to order at least 5 at a time. However, he also wants to maximize the number of orders he makes, so the skewers are as hot as possible. Find the number of ways he can order. (Note: the order of the orders is important.)

3. Max is a professional *Tetris* player. He notices that whenever he starts the game, the first seven pieces that fall is always a permutation of the seven pieces shown below. However, the purple piece always falls before the red one, the dark blue piece always falls before the yellow one, and the green piece always falls before the purple one. Under these conditions, how many possible orderings of the seven pieces are possible?



4. Vincent must take two tests for his fluid mechanics class, both of which have 25 problems, each worth 1 point. The first test is worth 40% of his grade, while the second is worth 60% of his grade. Find the minimum total of problems he must solve on both tests to have above an 80% average in the class.

5. Let $ABCD$ be a unit square, and construct equilateral triangle $\triangle ACE$ in its plane. Determine $(BE + DE)^2$.

6. Jeffrey is thinking of a positive integer N. He observes that $N - 1$ has 4 positive divisors and that N has 5 positive divisors. What is the sum of all possible values of N?

7. For a positive integer m, there are $150,000$ positive integers n such that mn has m digits. Find m.

8. For any integer $n > 1$, the function $f(n)$ outputs the smallest prime that does not divide n. Find $f(2) + f(3) + \ldots + f(100)$.

9. Victoria writes the fraction
$$\frac{1}{823544} = \frac{1}{7^7 + 1}$$
on a whiteboard. She performs operations on this number, which consists of adding 1 to the numerator and denominator and reducing the fraction to lowest terms. How many operations will she perform until $\frac{2019}{2020}$ is on the board?

10. Consider a convex quadrilateral $ABCD$ satisfying $\angle BAD + 2\angle BCD = 180^\circ$. Let P be the foot of the perpendicular from C to segment $\overline{BD}$. Suppose that $AB = 26$, $AD = 40$, $BP = 28$, and $PD = 14$. Find the area of $ABCD$.

2 Hints

1. Factor out a 3 and pair up terms.
2. How many orders does Sam make? If he order five each time, how many more does he need to order?
3. How many total ways are there? What fraction of these don't satisfy the condition?
4. Which test is more important?
5. Note that $BE + DE = 2EO$, if O is the center of $ABCD$.
6. What kind of numbers N have 5 positive divisors? When do these satisfy that $N - 1$ have 4 positive divisors?
7. Use the size of the number $150,000$ to estimate the size of m, and then check if you're right by calculating the number yourself.
8. Bound the size of $f(n)$. Use casework on its value.
9. When does the fraction reduce? How many moves does it take to happen?
10. Notice that C is the A-excenter of triangle $\triangle ABD$.

3 Answers

1. 680403
2. 120
3. 420
4. 38
5. 6
6. 16
7. 6
8. 286
9. 2060
10. 1386

4 Solutions

1. We can first take out a factor of 3 from each term, to get

$$3 + 6 + 9 + 12 + \cdots + 2019 = 3(1 + 2 + \cdots + 673).$$

Then, to compute the remaining sum, we can pair up 1 with 673, 2 with 672, and so on. There are 336 pairs summing to 674 and one singleton 337. Therefore, our total sum is

$$3 \cdot (336 \cdot 674 + 337) = \boxed{680403}.$$

Proposed by William Yue

2. To maximize the number of orders, Sam must make 8 orders. He cannot make 9 orders because then he would have to order at least $9 \cdot 5 = 45$ lamb skewers. If we put 5 skewers into each order, we have 3 remaining skewers to distribute among the 8 orders. We can think of this as 3 objects and 7 bars dividing them into 8 groups, which can be arranged in $\binom{10}{3} = \boxed{120}$ ways.

Proposed by Vincent Fan

3. Out of all 7! possible orderings, exactly one sixth of them satisfy that the three pieces green, purple, and red fall in that order. Additionally, exactly one half of these orderings satisfy that the dark blue piece falls before yellow piece. Therefore, the answer is

$$\frac{1}{6} \cdot \frac{1}{2} \cdot 7! = \boxed{420}.$$

Proposed by William Yue

4. Solving a problem on the first test is worth 1.6% of his total grade, and solving a problem on the second test is worth 2.4% of his total grade. Therefore, to get an 80%, he wants to first solve all the problems on the second test, before moving on to the first test. The 25 problems of the second test are worth $25 \cdot 2.4 = 60\%$ of his grade. For the remaining 20%, he needs to solve at least an additional 13 problems. Therefore, he must solve $25 + 13 = \boxed{38}$ problems in total.

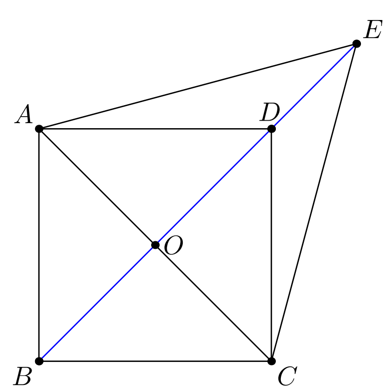
Proposed by Vincent Fan

5. Let O be the center of $ABCD$. Now note that

$$BE + DE = BO + OD + 2DE = 2(OD + DE) = 2OE.$$

However, since $AC = \sqrt{2}$ as ABC is a 45-45-90 triangle, we have $OC = \frac{\sqrt{2}}{2}$. Now, as OCE is a 30-60-90 triangle, we have that $OE = \frac{\sqrt{6}}{2}$. Hence

$$(BE + DE)^2 = (2OE)^2 = \boxed{6}.$$



Proposed by William Yue

6. It is well known that if the prime factorization of N is

$$N = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}.$$

then N has $(e_1 + 1)(e_2 + 1) \dots (e_k + 1)$ positive divisors.

Thus, we have

$$(e_1+1)(e_2+1)\dots(e_k+1)=5.$$

Since 5 is prime, we must have $e_1 = 4$ and all other powers equal to 0. Thus, N is of the form p^4 for some prime p.

Now, note that

$$N-1=p^4-1=(p^2-1)(p^2+1)=(p-1)(p+1)(p^2+1).$$

So, if $p-1 \neq 1$, then p^4 has at least 5 positive divisors. This is because $1, p-1, p+1, p^2+1, p^4$ all divide p^4, and no other pairs of numbers can be equal. This, however, is a contradiction since $N-1$ has 4 factors. Thus, we must have $p-1=1$, or $p=2$. Testing this, we see that it works, so $N = \boxed{16}$ is our only solution.

Proposed by Nathan Xiong

7. Note that there are exactly $10^m - 10^{m-1} = 9 \cdot 10^{m-1}$ positive integers with m digits, and approximately $1/m$ of them are divisible by m (specifically, in every m consecutive integers there is one that is divisible by m). Then the number of n such that mn has m digits is approximately

$$\frac{9 \cdot 10^{m-1}}{m}.$$

We note that $150,000$ is on the order of 10^5, so we expect 10^{m-1} to be on the same order, meaning that $m = 6$. Indeed, if $m = 6$, then there are 150,000 integers n such that $6n$ has 6 digits. To see this, note that $6 \cdot 16,667 = 100,002$ is the smallest number and $6 \cdot 166,666 = 999,998$ is the largest number, for a total of $166,666 - 16,667 + 1 = 150,000$ numbers. Thus our answer is $\boxed{6}$.

Proposed by Sebastian Zhu

8. Firstly, it's obvious that for all odd $n > 1$, we have $f(n) = 2$. Now, since $n \le 100$, $f(n)$ will never exceed 7 since $2 \times 3 \times 5 \times 7 = 210 > 100$.

Consider our even numbers between 2 and 100 inclusive. There are three possible cases:

Case 1: $f(n) = 3$ In this case, n is not divisible by 3. Note that there are 50 even numbers, and $\lfloor \frac{99}{6} \rfloor = 16$ of them are multiples of 3. So, for $50 - 16 = 34$ of our numbers n, we have $f(n) = 3$.

Case 2: $f(n) = 5$ In this case, n is divisible by 3 but not by 5. There are 16 multiples of 3, and $\lfloor \frac{16}{5} \rfloor = 3$ of these are multiples of 5. So, for $16 - 3 = 13$ of our numbers n, we have $f(n) = 5$.

Case 3: $f(n) = 7$ In this case, n is divisible by 3 and 5 but not by 7. Since $n < 100$, it suffices to find the number of even numbers which are multiples of 3 and 5. That is, the number of multiples of $2 \times 3 \times 5 = 30$. Clearly, there are 3 such numbers. So, for 3 of our numbers n, we have $f(n) = 7$.

Now, since there are 49 odd numbers between 2 and 100 inclusive, our answer is

$$49 \times 2 + 34 \times 3 + 13 \times 5 + 3 \times 7 = 98 + 102 + 65 + 21 = \boxed{286}.$$

Proposed by Nathan Xiong

9. It's important to notice that $\gcd(k, 7^n + k) = \gcd(k, 7^n)$ by the Euclidean algorithm. This means that the fraction

$$\frac{k}{7^n + k}$$

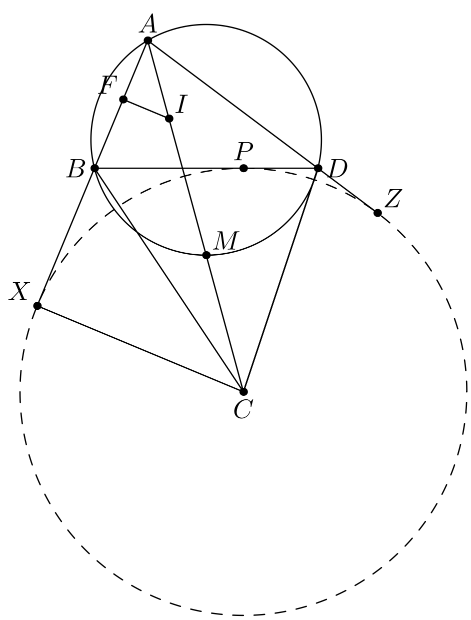
is reducible if and only if k is a multiple of 7. Starting from $k = 1$ and $n = 7$, after performing the operation 6 times we arrive at $\dfrac{7}{7^7 + 7} = \dfrac{1}{7^6 + 1}$. Now we can repeat the same argument with $n = 6$, and after 6 more operations our number reduces to $k = 1$ and $n = 5$. Our argument only fails when $n = 0$, since then 7^n is no longer divisible by 7. After 42 moves, we will finally have reduced our fraction to

$$\frac{1}{7^0 + 1} = \frac{1}{2}.$$

Finally, noting that the fraction $\dfrac{n-1}{n}$ is never reducible by the Euclidean algorithm, it will take us another 2018 operations to increase the numerator and denominator to 2019 and 2020, respectively. This means that the total number of operations used is $42 + 2018 = \boxed{2060}$.

Proposed by Sebastian Zhu

10. The key claim is that C is the A-excenter of $\triangle ABD$.



The A-excenter is defined to be the intersection of the A-angle bisector and the external angle bisectors of $\angle B$ and $\angle D$. It happens to be tangent to the segment BD, and the extensions of AB and AD past B and D respectively. Let X and Z be the associated tangency points on rays AB and AD. Define P' to be the excircle tangency point on $\overline{BD}$.

We will first prove that $AX = AZ = s$, where s is the semi perimeter of $\triangle ABD$. By equal tangents, we have $AX = AZ$. Then, by equal tangents again, we have

$$AX + AZ = AB + BP' + P'D + DA = AB + BD + DA = 2s.$$

Thus, we must have $AX = AZ = s$. It then follows that $BP' = BX = s - d$ and $DP' = DZ = s - b$. Now, we can easily compute that $s = 54$. Then, we get that $s - d = 28 = BP$ and $s - b = 14 = PD$.

Thus, $P = P'$.

Now, let C' be the A-excenter of $\triangle ABD$. Also, let M be the intersection of AC' with (ABD).

By the well known Incenter-Excenter Lemma, which is easily proven by angle chasing, M is the circumcenter of $\triangle BDC'$. Thus, using properties of cyclic quadrilaterals, we get that

$$\angle BAD + 2\angle BC'D = \angle BAD + \angle BMD = 180^\circ.$$

Now, it's obvious that there's only one point on the line through P perpendicular to $\overline{BD}$ satisfying this property and ensuring $ABCD$ is convex. Hence, $C = C'$.

Now, we can finish up the problem. We wish to compute CP, which is the radius of the A-excircle. The formula for the exradius is well known, but for the sake of completeness, we include its proof. Let I be the incenter of $\triangle ABD$, and F be the foot from I to $\overline{AB}$.

First, by AA similarity, we have $\triangle AFI \sim \triangle AXC$. Now, it's easy to show that $AF = s - a$. Thus, we have

$$\frac{AF}{FI} = \frac{AX}{XC} \iff \frac{s-a}{r} = \frac{s}{PC}.$$

Solving, we get that

$$PC = \frac{rs}{s-a} = \frac{[ABD]}{s-a}.$$

Applying Heron's Formula to $\triangle ABD$, we get that $[ABD] = 504$. And, $s - a = 12$. Hence, $PC = 42$.

Finishing off,

$$[ABCD] = [ABD] + [BCD] = 504 + \tfrac{42\times 42}{2} = 504 + 882 = \boxed{1386}.$$

Proposed by Nathan Xiong