MOAA 2019 Gunga Bowl

MOAA 2019: Gunga Bowl, Set 1

1. **[11]** Farmer John has 4000 gallons of milk in a bucket. On the first day, he withdraws 10% of the milk in the bucket for his cows. On each following day, he withdraws a percentage of the remaining milk that is 10% more than the percentage he withdrew on the previous day. For example, he withdraws 20% of the remaining milk on the second day. How much milk, in gallons, is left after the tenth day?

2. **[11]** Will multiplies the first four positive composite numbers to get an answer of w. Jeremy multiplies the first four positive prime numbers to get an answer of j. What is the positive difference between w and j?

3. **[11]** In Nathan's math class of 60 students, 75% of the students like dogs and 60% of the students like cats. What is the positive difference between the maximum possible and minimum possible number of students who like both dogs and cats?

MOAA 2019: Gunga Bowl, Set 2

4. **[13]** For how many integers x is $x^4 - 1$ prime?

5. **[13]** Right triangle $\triangle ABC$ satisfies $\angle BAC = 90^\circ$. Let D be the foot of the altitude from A to BC. If $AD = 60$ and $AB = 65$, find the area of $\triangle ABC$.

6. **[13]** Define $n! = n \times (n-1) \times \cdots \times 1$. Given that

$$3! + 4! + 5! = a^2 + b^2 + c^2$$

for distinct positive integers a, b, c, find $a + b + c$.

MOAA 2019: Gunga Bowl, Set 3

7. **[15]** Max nails a unit square to the plane. Let M be the number of ways to place a regular hexagon (of any size) in the same plane such that the square and hexagon share at least 2 vertices. Vincent, on the other hand, nails a regular unit hexagon to the plane. Let V be the number of ways to place a square (of any size) in the same plane such that the square and hexagon share at least 2 vertices. Find the nonnegative difference between M and V.

8. **[15]** Let a be the answer to this question, and suppose $a > 0$. Find $\sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}$.

9. **[15]** How many ordered pairs of integers (x, y) are there such that $x^2 - y^2 = 2019$?

MOAA 2019: Gunga Bowl, Set 4

10. **[17]** Compute

$$\frac{p^3 + q^3 + r^3 - 3pqr}{p + q + r}$$

where $p = 17$, $q = 7$, and $r = 8$.

11. **[17]** The unit squares of a 3×3 grid are colored black and white. Call a coloring *good* if in each of the four 2×2 squares in the 3×3 grid, there is either exactly one black square or exactly one white square. How many good colorings are there? Consider rotations and reflections of the same pattern distinct colorings.

12. **[17]** Define a *k-respecting string* as a sequence of k consecutive positive integers $a_1, a_2, \ldots, a_k$ such that a_i is divisible by i for each $1 \leq i \leq k$. For example, $7, 8, 9$ is a 3-respecting string because 7 is divisible by 1, 8 is divisible by 2, and 9 is divisible by 3. Let S_7 be the set of the first terms of all 7-respecting strings. Find the sum of the three smallest elements in S_7.

MOAA 2019: Gunga Bowl, Set 5

13. **[19]** A triangle and a quadrilateral are situated in the plane such that they have a finite number of intersection points I. Find the sum of all possible values of I.

14. **[19]** Mr. DoBa continuously chooses a positive integer at random such that he picks the positive integer N with probability 2^{-N}, and he wins when he picks a multiple of 10. What is the expected number of times Mr. DoBa will pick a number in this game until he wins?

15. **[19]** If a, b, c, d are all positive integers less than 5, not necessarily distinct, find the number of ordered quadruples (a, b, c, d) such that

$$a^b - c^d$$

is divisible by 5.

MOAA 2019: Gunga Bowl, Set 6

16. **[21]** Let $n! = n \times (n-1) \times \cdots \times 2 \times 1$. Find the maximum positive integer value of x such that the quotient

$$\frac{160!}{160^x}$$

is an integer.

17. **[21]** Let $\triangle OAB$ be a triangle with $\angle OAB = 90^\circ$. Draw points C, D, E, F, G in its plane so that

$$\triangle OAB \sim \triangle OBC \sim \triangle OCD \sim \triangle ODE \sim \triangle OEF \sim \triangle OFG,$$

and none of these triangles overlap. If points O, A, G lie on the same line, then let x be the sum of all possible values of $\frac{OG}{OA}$. Then, x can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m, n. Compute $m + n$.

18. **[21]** Let $f(x)$ denote the least integer greater than or equal to $x^{\sqrt{x}}$. Compute $f(1)+f(2)+f(3)+f(4)$.

MOAA 2019: Gunga Bowl, Set 7

The Fibonacci sequence $\{F_n\}$ is defined as $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all integers $n \geq 0$.

19. **[23]** Find the least odd prime factor of $(F_3)^{20} + (F_4)^{20} + (F_5)^{20}$.

20. **[23]** Let
$$S = \frac{1}{F_3F_5} + \frac{1}{F_4F_6} + \frac{1}{F_5F_7} + \frac{1}{F_6F_8} + \cdots$$
Compute $420S$.

21. **[23]** Consider the number
$$Q = 0.0001010203050801302103405508901 44\ldots,$$
the decimal created by concatenating every Fibonacci number and placing a 0 right after the decimal point and between each Fibonacci number. Find the greatest integer less than or equal to $\frac{1}{Q}$.

MOAA 2019: Gunga Bowl, Set 8

22. **[26]** In five dimensional hyperspace, consider a hypercube C_0 of side length 2. Around it, circumscribe a hypersphere S_0, so all 32 vertices of C_0 are on the surface of S_0. Around S_0, circumscribe a hypercube C_1, so that S_0 is tangent to all hyperfaces of C_1. Continue in this same fashion for S_1, C_2, S_2, and so on. Find the side length of C_4.

23. **[26]** Suppose $\triangle ABC$ satisfies $AC = 10\sqrt{2}$, $BC = 15$, $\angle C = 45^\circ$. Let D, E, F be the feet of the altitudes in $\triangle ABC$, and let U, V, W be the points where the incircle of $\triangle DEF$ is tangent to the sides of $\triangle DEF$. Find the area of $\triangle UVW$.

24. **[26]** A polynomial $P(x)$ is called *spicy* if all of its coefficients are nonnegative integers less than 9. How many spicy polynomials satisfy $P(3) = 2019$?

The next set will consist of three estimation problems.

MOAA 2019: Gunga Bowl, Set 9

Points will be awarded based on the formulae below. Answers are nonnegative integers *that may exceed* 1,000,000.

25. **[30]** Suppose a circle of radius 20192019 has area A. Let s be the side length of a square with area A. Compute the greatest integer less than or equal to s.

    If n is the correct answer, an estimate of e gives $\max\left\{0, \left\lfloor 1030\left(\min\left\{\frac{n}{e}, \frac{e}{n}\right\}\right)^{18}\right\rfloor - 1000\right\}$ points.

26. **[30]** Given a 50×50 grid of squares, initially all white, define an *operation* as picking a square and coloring it and the four squares horizontally or vertically adjacent to it blue, if they exist. If a square is already colored blue, it will remain blue if colored again. What is the minimum number of operations necessary to color the entire grid blue?

    If n is the correct answer, an estimate of e gives $\left\lfloor \frac{180}{5|n-e|+6} \right\rfloor$ points.

27. **[30]** The sphere packing problem asks what percent of space can be filled with equally sized spheres without overlap. In three dimensions, the answer is $\frac{\pi}{3\sqrt{2}} \approx 74.05\%$ of space (confirmed as recently as 2017!), so we say that the *packing density* of spheres in three dimensions is about 0.74. In fact, mathematicians have found optimal packing densities for certain other dimensions as well, one being eight-dimensional space. Let d be the packing density of eight-dimensional hyperspheres in eight-dimensional hyperspace. Compute the greatest integer less than $10^8 \times d$.

    If n is the correct answer, an estimate of e gives $\max\left\{\lfloor 30 - 10^{-5}|n-e| \rfloor, 0\right\}$ points.