MOAA 2019 Accuracy Round Solutions

1. Let's say that a cow eats one unit of grass per day. By the first condition, the pasture provides $100 \cdot 10 = 1000$ units of grass over 10 days. By the second condition, the pasture provides $100 \cdot 5 + 120 \cdot 3 = 860$ units of grass over 8 days. Thus, it must be true that in two days, the pasture produces $1000 - 860 = 140$ units of grass. Therefore the pasture produces 70 units of grass per day, and so it is able to sustain 70 cows per day indefinitely. Any more cows than that would eat more than the pasture could produce, so eventually the pasture would run out of grass, and the answer is indeed $\boxed{70}$.

2. Reducing the equation modulo 3 gives
$$2 \pm 2 \pm 0 \pm 1 \pm 2 \pm 0.$$
However, note that the choice of ± 2 is equivalent to ± 1 in modulo 3. Hence we can replace this with
$$-1 \pm 1 \pm 0 \pm 1 \pm 1 \pm 0.$$
Now note that we must have two $+1$s and one -1. This can happen in 3 ways. Then we have 2 choices for each of the two 0s, giving an answer of $3 \cdot 4 = \boxed{12}$.

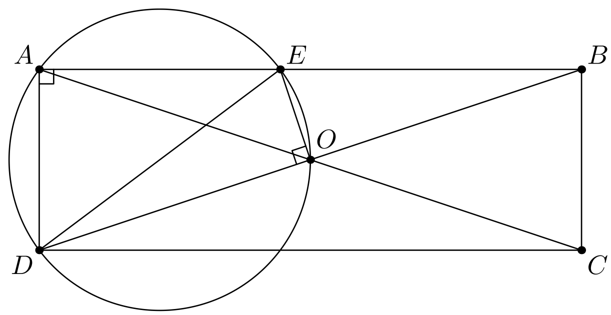
3. We apply complementary counting. First we count the total number of rectangles, then we subtract off the number that contain both red squares. The key observation is that each rectangle is defined by four lines: two horizontal and two vertical. Therefore, to count the total number of rectangles, we multiply the number of ways to choose two horizontal lines by the number of ways to choose two vertical lines:
$$\binom{7}{2} \cdot \binom{6}{2} = 315.$$
We can count the number of rectangles that contain both red squares in a similar way. There are 2 ways to choose both the top and bottom horizontal lines, and 2 ways to choose the left vertical line but 3 ways to choose the right vertical line. (These measurements ensure that a rectangle we choose will contain both red squares.) In total, this gives
$$2 \cdot 2 \cdot 2 \cdot 3 = 24$$
rectangles that contain both red squares. Therefore, the number of rectangles with at most one red square is
$$315 - 24 = \boxed{291}.$$

4. Let $P(x) = ax^3 + bx^2 + cx + d$. Note that we want to find $\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1} = \frac{r_1 + r_2 + r_3}{r_1 r_2 r_3}$, and that $r_1 + r_2 + r_3 = -\frac{b}{a}$ and $r_1 r_2 r_3 = -\frac{d}{a}$ by Vieta's formulas. Thus we are really looking for the value of $\frac{b}{d}$. We are given that
$$\frac{(-8a + 4b - 2c + d) + (8a + 4b + 2c + d)}{d} = 200.$$
Canceling and simplifying yields
$$\begin{aligned} 200 &= \frac{8b + 2d}{d} \\ 100d &= 4b + d \\ \frac{b}{d} &= \frac{99}{4}. \end{aligned}$$
Our answer is thus $99 + 4 = \boxed{103}$.

5. Note that since $\angle DAE = 90^\circ$, $\overline{ED}$ is a diameter of the circle. Thus $\angle EOD = 90^\circ$ as well. Now, observe that $\angle EOB = \angle ADC = 90^\circ$ and $\angle EBO = \angle ACD$ since we are in a rectangle. Thus $\triangle EOB \sim \triangle ADC$ by AA similarity. By the Pythagorean Theorem, $BD = AC = \sqrt{3^2+1^2} = \sqrt{10}$. Thus $OB = \frac{\sqrt{10}}{2}$. From similar triangles, we have

$$EB = AC \cdot \frac{OB}{DC} = \sqrt{10} \cdot \frac{\sqrt{10}}{6} = \frac{5}{3}.$$

Our answer is thus $\boxed{8}$.



6. Extend AD past D to a point A' such that $AD = DA'$. Note that A' is the point opposite A on the circle centered at D with radius 100. Thus $\angle AEA' = 90^\circ$. Also, since A and B are opposite to each other on the circle with center M with radius 50, $\angle AEB = 90^\circ$. This means that B, E, A' are collinear.

Now call $FB = x$ and extend FC past C to a point G such that $A'G \parallel DC$. Note that $AFGA'$ is a trapezoid with midline DC. Furthermore, since $AF = 100 - x$ and $DC = 100$, it must be true that $A'G = 100 + x$. In addition, note that by AA similarity, $\triangle AEB \sim \triangle A'EA \sim \triangle A'AB$. Thus

$$\frac{BE}{EA} = \frac{BA}{AA'} = \frac{1}{2}$$

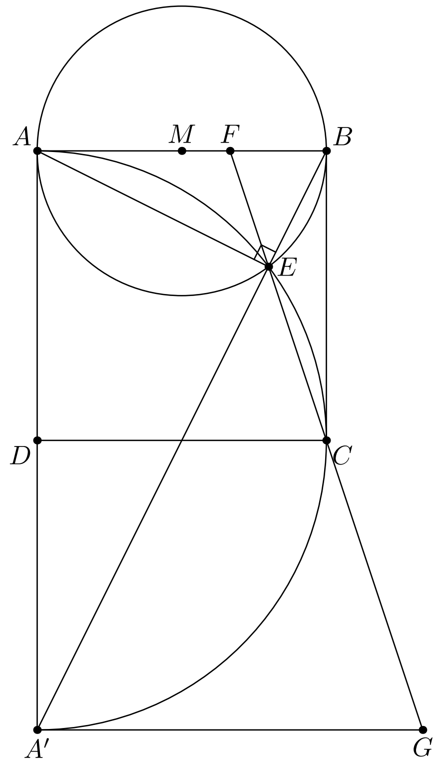
and

$$\frac{A'E}{EA} = \frac{A'A}{AB} = 2.$$

Thus $4BE = 2EA = A'E$. In addition, we may note that by construction, $\triangle FBE \sim \triangle GA'E$. Thus $A'G = FB \cdot \frac{BE}{EA'} = 4x$. We now have two expressions for the length of $A'G$, and they must be equal:

$$4x = 100 + x \implies x = \frac{100}{3}.$$

Thus $AF = 100 - \frac{100}{3} = \frac{200}{3}$, and the answer is $\boxed{203}$.



7. If $3x + 5y$ and $5x + 2y$ are both integers, then so are linear combinations of them. In other words, if we multiply both expressions by an integer and then subtract them, we should still get an integer. Thus, if we multiply the first expression by 2 and the second expression by 5 and subtract, we get $5(5x + 2y) - 2(3x + 5y) = 19x$. This number must still be an integer; thus, x must be of the form $\frac{a}{19}$ for some integer $0 < a < 19$. Similarly, if we multiply the first equation by 5 and the second equation by 3 and subtract, we get $5(3x + 5y) - 3(5x + 2y) = 19y$. Since this must be an integer, y must be of the form $\frac{b}{19}$ for some integer $0 < b < 19$.

   Thus we have reduced our problem to finding $0 < a, b < 19$ such that $3a + 5b \equiv 5a + 2b \equiv 0 \pmod{19}$. Note first that if $3a + 5b \equiv 0 \pmod{19}$, then $0 \equiv 8(3a + 5b) \equiv 24a + 40b \equiv 5a + 2b \pmod{19}$. Thus it suffices to determine all a, b such that 19 divides $3a+5b$. However, for each $0 < a < 19$, we may choose the residue $b \equiv 3a \cdot 5^{-1} \equiv 12a \pmod{19}$, where $5^{-1} = 4$ denotes the inverse of $5 \pmod{19}$. Therefore, for each $0 < a < 19$, there is exactly one value of b such that 19 divides $3a+5b$. There are thus 18 pairs of integers (a, b) between 0 and 19 exclusive such that 19 divides $3a+5b$ and $5a+2b$, and consequently $\boxed{18}$ pairs of real numbers (x, y) between 0 and 1 exclusive such that $3x+5y$ and $5x+2y$ are both integers.

   Alternatively, one may find the answer to this problem by graphical methods, despite the approach being non-rigorous.

8. Let's do casework on the number of sections S at the end. Note that there are a total of $4^4 = 256$ ways to color the spinner.

   - If $S = 1$, all four sections of the spinner must be colored the same. There are clearly 4 ways to do this, so the probability of this occurring is $\frac{4}{256}$.
   - If $S = 2$, we have a couple of cases. Note that we cannot have more than two colors since that would result in at least 3 sections.
     - There are three sections of one color and one of another. There are $4 \cdot 3 = 12$ ways to choose the colors, and 4 ways to arrange them, resulting in 48 cases.

  - There are two sections of one color and two of another. There are $\binom{4}{2} = 6$ ways to choose the colors. The sections of the same color cannot be across from each other, so they must be touching. This gives 4 ways to arrange them, resulting in 24 cases.

  In total, there are 72 arrangements, so the probability of this occurring is $\frac{72}{256}$.

- If $S = 3$, note that we cannot have four colors since that would result in 4 sections. We also cannot have two colors since there is no way to arrange them into three sections. Therefore, there are two adjacent sections of one color, and the two remaining sections are two other colors. There are $4 \cdot 3 = 12$ ways to choose these colors, and $4 \cdot 2 = 8$ ways to arrange them, resulting in 96 arrangements, so the probability of this occurring is $\frac{96}{256}$.
- If $S = 4$, we have many cases:
  - There are four different colors. This results in 24 cases.
  - There are three different colors. The one that appears twice must have its sections opposite each other. There are $4 \cdot 3 = 12$ ways to choose the colors and $2 \cdot 2 = 4$ ways to arrange them, resulting in 48 cases.
  - There are two different colors. Then they must both be opposite each other in a checkerboard configuration. There are $\binom{4}{2} = 6$ ways to choose the colors and 2 ways to arrange them, resulting in 12 cases.

  In total, there are 84 arrangements, so the probability of this occurring is $\frac{84}{256}$.

We can quickly check that these probabilities sum to 1, so we haven't made an obvious mistake. The expected value of S is then

$$\frac{1}{256}\left(1 \cdot 4 + 2 \cdot 72 + 3 \cdot 96 + 4 \cdot 84\right) = \frac{193}{64},$$

and our answer is $193 + 64 = \boxed{257}$.

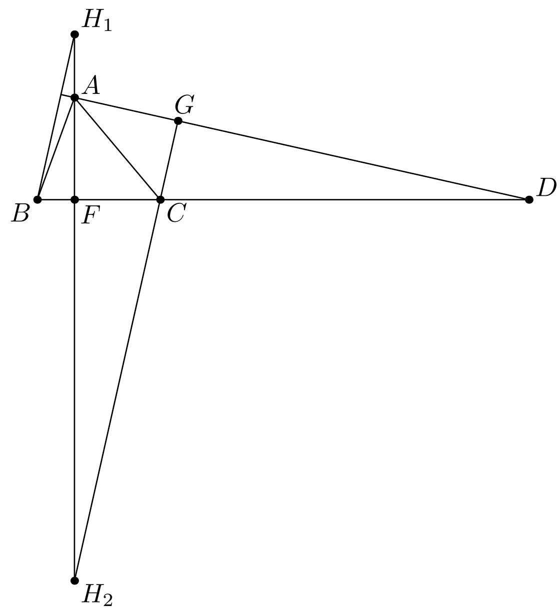
9. Let F be the foot of the perpendicular from A to BC and G be the foot of the perpendicular from C to AD. Note that $AF = 12$, $BF = 5$, and $CF = 9$. Since CH_2 and BH_1 are parallel, $\triangle BH_1F$ is similar to $\triangle CH_2F$. Therefore,

$$\frac{H_1F}{FH_2} = \frac{BF}{FC} = \frac{5}{9}.$$

However, since $H_1F + FH_2 = H_1H_2 = 1001$, we know that $H_2F = \frac{9}{14} \cdot H_1H_2 = \frac{1287}{2}$. Now note that since $\angle FCH_2 = \angle GCD = 90^\circ - \angle CDG = \angle FAD$ and $\angle CFH_2 = \angle AFD = 90^\circ$, $\triangle CFH_2$ is similar to $\triangle AFD$. Then,

$$\frac{AF}{FD} = \frac{CF}{FH_2} = \frac{9}{1287/2} = \frac{2}{143}.$$

Since $AF = 12$, $FD = 858$, so $CD = 858 - 9 = \boxed{849}$.



10. Rewrite $k \cdot (k-1) \cdots 2 \cdot 1$ as $k!$. The key is noting that we can rewrite

$$\frac{n+1}{n!} = \frac{1}{n!} + \frac{1}{(n-1)!}.$$

Using this, we can telescope our sum:

$$\begin{aligned} \frac{2}{1!} - \frac{3}{2!} + \frac{4}{3!} + \cdots + (-1)^{k+1}\frac{k+1}{k!} &= \frac{1}{0!} + \frac{1}{1!} - \frac{1}{1!} - \frac{1}{2!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{(-1)^{k+1}}{(k-1)!} + \frac{(-1)^{k+1}}{k!} \\ &= 1 + \frac{(-1)^{k+1}}{k!}, \end{aligned}$$

where everything but the first and last term cancels. Thus, we must have

$$1 + \frac{(-1)^{k+1}}{k!} \geq 1 + \frac{1}{700^3} \implies \frac{(-1)^{k+1}}{k!} \geq \frac{1}{700^3}.$$

Clearly, k must be odd, as otherwise the left hand side would be negative while the right hand side would be positive. Thus, we are interested in the positive odd integer solutions to

$$\frac{1}{k!} \geq \frac{1}{700^3} \implies k! \leq 700^3.$$

Now, note that $12! > 700^3$. Indeed:

$$12! = 6! \cdot (8 \cdot 9 \cdot 10) \cdot (7 \cdot 11 \cdot 12) = 720^2 \cdot 924 > 700^3.$$

But, for all $k \leq 11$, we have $k! < 700^3$. Indeed:

$$11! = 5! \cdot (7 \cdot 9 \cdot 10) \cdot (11 \cdot 8 \cdot 6) = 120 \cdot 630 \cdot 528 < 700^3.$$

The sum of the 6 odd numbers between 1 and 11 inclusive is $6^2 = \boxed{36}$.