MOAA 2019 Gunga Bowl Solutions

MOAA 2019 Gunga Bowl Set 1

1. On the tenth day, Farmer John withdraws 100% of the remaining milk from the bucket. Therefore, the answer is $\boxed{0}$.

2. The product of the first four composite positive integers is
$$w = 4 \cdot 6 \cdot 8 \cdot 9 = 1728,$$
and the product of the first four prime positive integers is
$$j = 2 \cdot 3 \cdot 5 \cdot 7 = 210.$$
The difference between these two number is $1728 - 210 = \boxed{1518}$.

3. The maximum percentage of people who like both dogs and cats is 60%. Obviously we can't have more than this, since only 60% of people like cats, and this is attainable when everyone who likes cats also likes dogs.

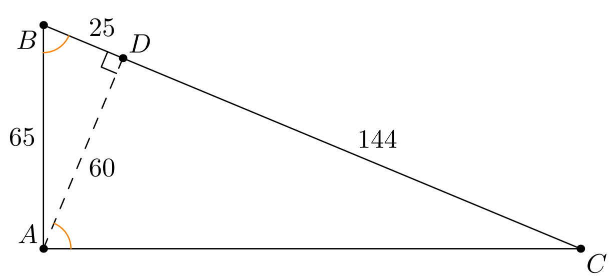
   The minimum percentage of people who like both dogs and cats is 35%. We can't have less than this because then the percentage of people who like either dogs *or* cats would be more than $65+70-35 = 100\%$. This is attainable when, of the 65% of remaining people, 40% like only dogs and 25% like only cats.

   Therefore, the maximum number of people is 36 and the minimum number of people is 21. Their difference is $36 - 21 = \boxed{15}$.

MOAA 2019 Gunga Bowl Set 2

4. We can factor the expression as $x^4 - 1 = (x^2+1)(x^2-1)$. Note that $x^2+1 > 0$, so in order for this to be prime, one of the factors must be equal to 1. $x^2+1 = 1$ gives $x = 0$, which doesn't work. $x^2 - 1 = 1$ gives no integer solutions. Therefore there are $\boxed{0}$ solutions.

5. Observe that $\triangle ABD$ is a $5-12-13$ triangle, so $BD = 5 \cdot 5 = 25$. (Alternatively, the Pythagorean Theorem works just fine.) Now, noting that $\angle ABD = \angle DAC$ since they are both complementary to $\angle BAD$, we know that $\triangle ABD \sim \triangle CAD$ by AA similarity (note that $\angle BDA = \angle ADC = 90°$). Thus
$$\frac{BD}{DA} = \frac{AD}{DC} \implies BD \cdot DC = AD^2.$$
Then $DC = \frac{60^2}{25} = 12^2 = 144$. Finally, the area of $\triangle ABC$ is half base times height, or
$$\frac{1}{2} \cdot AD \cdot BC = \frac{1}{2} \cdot 60 \cdot (25+144) = \boxed{5070}.$$



6. Write

$$3! + 4! + 5! = (6 - 2) + (24 + 1) + (120 + 1) = 2^2 + 5^2 + 11^2,$$

so the answer is $2 + 5 + 11 = \boxed{18}$. Alternatively, one can write out the sum $6 + 24 + 120 = 150$, and subsequently notice that $150 = 100 + 49 + 1 = 10^2 + 7^2 + 1^2$. The sum in this case is also 18. There is one more set of three squares that sum to 150, namely $10^2 + 5^2 + 5^2$; however, this set violates the distinctness condition.

MOAA 2019 Gunga Bowl Set 3

7. It is clear that a square and a regular hexagon can share at most 2 vertices. For Max, he just needs to chose two vertices of the nailed square in $\binom{4}{2}$ ways, and then he can attach a hexagon to it. Visibly he can attach it in 5 ways, giving a total of

$$M = \binom{4}{2} \cdot 5 = 30.$$

For Vincent, he chooses two vertices of the nailed hexagon in $\binom{6}{2}$ ways, and then he can attach a square to it. It is also clear that he can attach the square on in 3 ways, giving a total of

$$V = \binom{6}{2} \cdot 3 = 45.$$

The final answer is $45 - 30 = \boxed{15}$.

Some may ask conceptually why the answer isn't 0. The answer is due to rotations: on Vincent's hexagon, he has more ways to rotate the same diagram to get different configurations. This way of thinking is corroborated when we notice that V is exactly $\frac{6}{4}$ times M.

8. We have

$$\begin{aligned} a &= \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}} \\ a &= \sqrt{a + a} \\ a^2 &= 2a \\ a &= \boxed{2}, \end{aligned}$$

since $a \neq 0$.

Although the following is not expected of contestants, it is possible to prove convergence of the square roots in the problem by the monotone convergence theorem. There are two steps: showing that every element in the sequence defined by $a_1 = \sqrt{a}$ and $a_{k+1} = \sqrt{a + a_k}$ for positive integers k is less than a, and showing that the sequence is monotonically increasing, both via induction.

9. We can factor

$$(x - y)(x + y) = x^2 - y^2 = 2019 = 3 \cdot 673.$$

Then we can note that given $x - y = a$ and $x + y = b$, we can solve this for $x = \frac{a+b}{2}$ and $y = \frac{b-a}{2}$, so each pair (a, b) corresponds to a unique integer solution (x, y), as long as a and b have the same parity. We can factor 2019 into two ordered positive factors in four ways:

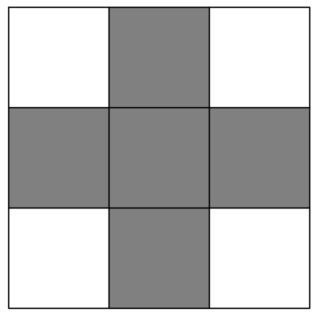
$$2019 = 1 \cdot 2019 = 3 \cdot 673 = 673 \cdot 3 = 2019 \cdot 1.$$

However, there are the same number of factorings into two ordered negative factors as well. Since in all these cases the two factors have the same parity, our answer is $\boxed{8}$.

MOAA 2019 Gunga Bowl Set 4

10. We can factor
$$p^3 + q^3 + r^3 - 3pqr = (p + q + r)(p^2 + q^2 + r^2 - pq - qr - rp),$$
so we just need to compute
$$p^2 + q^2 + r^2 - pq - qr - rp = 289 + 49 + 64 - 119 - 136 - 56 = \boxed{91}.$$

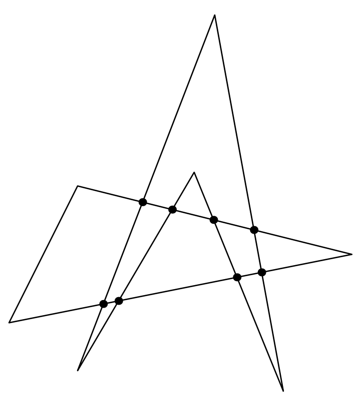
11. Note that, given any 2×2 grid with 3 of the 4 entries filled in, the last entry is always fixed. Thus we can color in the five squares forming a cross in the 3×3 grid (shaded below) in any fashion, and this coloring fixes the four corner squares. There are $2^5 = \boxed{32}$ ways to color in the five cross squares.



12. Suppose our 7-respecting sequence is
$$x, x + 1, x + 2, x + 3, x + 4, x + 5, x + 6.$$
We need $1 \mid x$, $2 \mid x + 1$, $3 \mid x + 2$, $4 \mid x + 3$, $5 \mid x + 4$, $6 \mid x + 5$, $7 \mid x + 6$. However, this is equivalent to needing $1 \mid x - 1$, $2 \mid x - 1$, $3 \mid x - 1$, $4 \mid x - 1$, $5 \mid x - 1$, $5 \mid x - 1$, $7 \mid x - 1$ by subtracting the divisor from each of the dividends. Thus, it is necessary and sufficient for $x - 1$ to be divisible by all of $1, 2, 3, 4, 5, 6, 7$. Then $x - 1$ must be a multiple of 420, so that $x = 1, 421, 841$ are the three smallest possible values of x. These sum to the answer of $\boxed{1263}$.

MOAA 2019 Gunga Bowl Set 5

13. We claim that the only possible values of I are the integers between 0 and 8, inclusive, giving a total of $\boxed{36}$. The case for 8 intersection points is given below, and it's easy to see how to adjust this for smaller cases.



Now we claim that 9 or more intersections is impossible. By the Pigeonhole Principle, this would imply that one side of the quadrilateral is intersected by all three sides of the triangle. However, this is impossible: if our triangle is ABC, then since AB intersects a side of the quadrilateral, A, B are on opposite sides of that side. Similarly, B, C and C, A are on opposite sides of that side, a contradiction since A cannot be on the opposite side to itself.

14. The probability that Mr. DoBa picks a multiple of 10 during any given pick is

$$\begin{aligned} p &= 2^{-10} + 2^{-20} + 2^{-30} + \cdots \\ &= \frac{2^{-10}}{1-2^{-10}} \\ &= \frac{1}{2^{10}-1} \\ &= \frac{1}{1023}. \end{aligned}$$

Experienced contestants know that for an event that occurs with a constant probability p, the expected number of times before the event occurs is $\frac{1}{p}$. We go through the derivation below.

The probability that Mr. DoBa wins in one turn is $\frac{1}{1023}$. The probability that he wins in two turns is $\frac{1022}{1023} \cdot \frac{1}{1023}$. Continuing in this way, the probability that he wins in exactly n turns is $\left(\frac{1022}{1023}\right)^{n-1} \cdot \frac{1}{1023}$. Thus, the expected number of turns he will take is, by definition,

$$E = 1 \cdot \frac{1}{1023} + 2 \cdot \frac{1022}{1023} \cdot \frac{1}{1023} + 3 \cdot \left(\frac{1022}{1023}\right)^2 \cdot \frac{1}{1023} + \cdots.$$

Letting $x = \frac{1022}{1023}$, we find

$$\begin{aligned} E - xE &= \frac{1}{1023}(1 + 2x + 3x^2 + \cdots - x - 2x^2 - 3x^3 - \cdots) \\ (1-x)E &= \frac{1}{1023}(1 + x + x^2 + \cdots) \\ (1-x)E &= \frac{1}{1023} \cdot \frac{1}{1-x} \\ E &= \frac{1}{1023} \cdot \frac{1}{(1-x)^2} \\ E &= \frac{1}{1023} \cdot 1023^2 \\ E &= \boxed{1023}. \end{aligned}$$

15. We can create a table in modulo 5 of all the possible values of x^y, where the different values of x run along the top and the different values of y run along the left side:

$x^y \pmod 5$	1	2	3	4
1	1	2	3	4
2	1	4	4	1
3	1	3	2	4
4	1	1	1	1

Note that 1 appears eight times, 2 appears two times, 3 appears two times, and 4 appears four times. Let us first consider the case where $a^b \equiv c^d \equiv 1 \pmod 5$. Then, they both have to be one of the 8 values, giving $8^2 = 64$ possibilities. Similarly, we can get $2^2 = 4$, $2^2 = 4$, and $4^2 = 16$ possibilities for the other cases. Hence, we get the answer

$$64 + 4 + 4 + 16 = \boxed{88}.$$

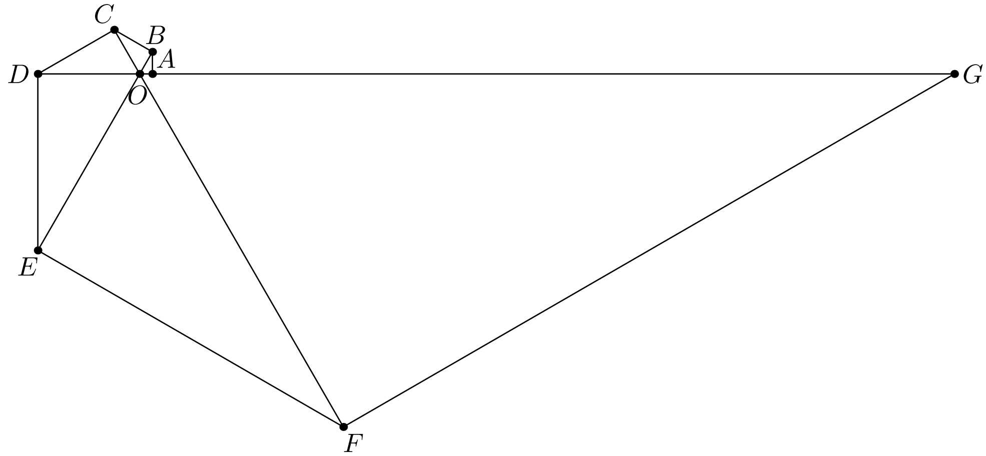
MOAA 2019 Gunga Bowl Set 6

16. Note that $160 = 2^5 \cdot 5$. We will thus count the number of factors of 2 and 5 in the numerator and determine the suitable x from that information.

The number of factors of 2 in the numerator is $80 + 40 + 20 + 10 + 5 + 2 + 1 = 158$ by Legendre's formula. (This adds one for each number divisible by 2, then one more for each number divisible by 4, then one more for each number divisible by 8, and so on.) Therefore, we require $5x \leq 158 \implies x \leq 31$. Also by Legendre's formula, the number of factors of 5 in the numerator is $32 + 6 + 1 = 39$. Thus we require $x \leq 39$. Then $x \leq 31$ is the limiting restriction, and the maximum possible value of x is $\boxed{31}$.

17. We have two cases:

- G is on the same side of O as A.

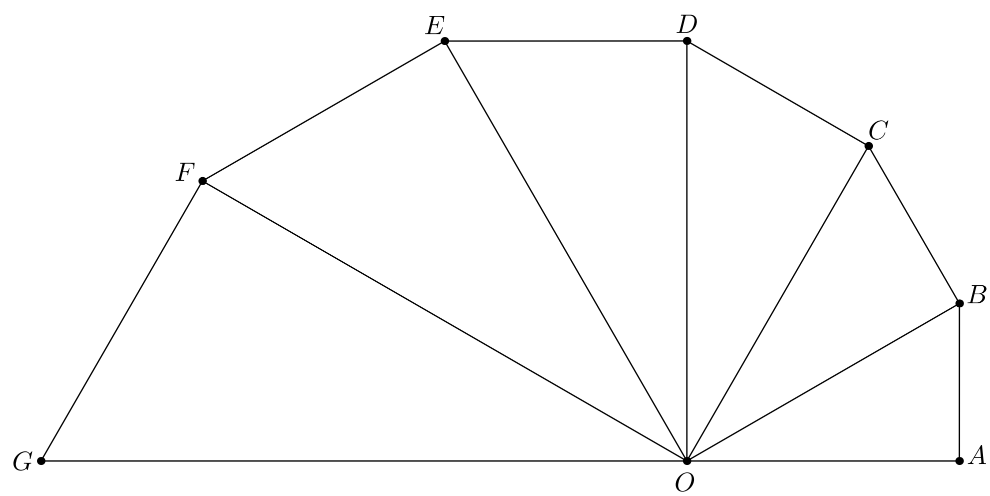


Because the triangles are similar, note that the 360 degrees of rotation must be evenly split by the six triangles. Therefore, $\angle AOB = 60^\circ$, and all six triangles are $30 - 60 - 90$ ones. Therefore

$$OG = 2OF = 4OE = 8OD = 16OC = 32OB = 64OA,$$

so $\frac{OG}{OA} = 64$.

- G is on the opposite side of O as A.



Now, the 180 degrees of rotation must be evenly split by the six triangles. Therefore, $\angle AOB = 30^\circ$, and we have six $30 - 60 - 90$ triangles again, but this time in a different orientation. In this case,

$$OG = \frac{2}{\sqrt{3}}OF = \frac{4}{3}OE = \frac{8}{3\sqrt{3}}OD = \frac{16}{9}OC = \frac{32}{9\sqrt{3}}OB = \frac{64}{27}OA,$$

so $\frac{OG}{OA} = \frac{64}{27}$.

Summing these two cases gives

$$64 + \frac{64}{27} = \frac{1792}{27},$$

which gives an answer of $1792 + 27 = \boxed{1819}$.

18. Let's compute each term separately. Let $\lceil x \rceil$ denote the least integer greater than or equal to x.

- $f(1) = \lceil 1^{\sqrt{1}} \rceil$, which equals 1.
- $f(2) = \lceil 2^{\sqrt{2}} \rceil$. Note that since $\sqrt{2} \approx 1.4$, we can bound $1 < \sqrt{2} < \frac{3}{2}$. (We can also just square all three terms to confirm this.) Therefore,

  $$2^1 < 2^{\sqrt{2}} < 2^{3/2}.$$

  However, since $8 < 9$, taking the square root of both sides gives $2^{3/2} < 3$. Therefore $2^{\sqrt{2}}$ is between 2 and 3, so its ceiling is 3.
- $f(3) = \lceil 3^{\sqrt{3}} \rceil$. We can do a similar bound by noting $\sqrt{3} \approx 1.7$, which gives $\frac{5}{3} < \sqrt{3} < \frac{7}{4}$. (Also can be confirmed by squaring all terms.) Therefore,

  $$3^{5/3} < 3^{\sqrt{3}} < 3^{7/4}.$$

  However, note that $6 < 3^{5/3}$ since $216 = 6^3 < 3^5 = 243$, and $3^{7/4} < 7$ since $2187 = 3^7 < 7^4 = 2401$. Therefore $3^{\sqrt{3}}$ is between 6 and 7, so $\lceil 3^{\sqrt{3}} \rceil$ is 7.
- $f(4) = \lceil 4^{\sqrt{4}} \rceil$, which equals 16.

Adding these four values gives $1 + 3 + 7 + 16 = \boxed{27}$.

MOAA 2019 Gunga Bowl Set 7

19. Firstly, $F_3 = 2$, $F_4 = 3$, and $F_5 = 5$. Recall Fermat's Little Theorem, which states that for a prime p and integer a relatively prime to p,

$$a^{p-1} \equiv 1 \pmod{p}.$$

Using this, we can begin to test primes. Note that it's fairly easy to test primes that are just smaller than 20, and indeed, we find that

$$2^{20} + 3^{20} + 5^{20} \equiv 2^2 + 3^2 + 5^2 \equiv 38 \equiv 0 \pmod{19}.$$

Thus we need only to test all odd primes less than 19 to see if there is a smaller one.

- When $p = 3$,

  $$2^{20} + 3^{20} + 5^{20} \equiv 2^0 + 0 + 5^0 \equiv 2 \pmod{3}.$$

- When $p = 5$,

  $$2^{20} + 3^{20} + 5^{20} \equiv 2^0 + 3^0 + 0 \equiv 2 \pmod{5}.$$

- When $p = 7$,

  $$2^{20} + 3^{20} + 5^{20} \equiv 2^2 + 3^2 + 5^2 \equiv 38 \equiv 3 \pmod{7}.$$

- When $p = 11$,

  $$2^{20} + 3^{20} + 5^{20} \equiv 2^0 + 3^0 + 5^0 \equiv 3 \pmod{11}.$$

- When $p = 13$,

  $$2^{20} + 3^{20} + 5^{20} \equiv 2^8 + 3^8 + 5^8 \equiv 3^2 + 3^2 + (-1)^4 \equiv 6 \pmod{13}.$$

- When $p = 17$,

  $$2^{20} + 3^{20} + 5^{20} \equiv 2^4 + 3^4 + 5^4 \equiv 16 + 81 + 64 \equiv 161 \equiv 8 \pmod{17}.$$

Therefore $2^{20}+3^{20}+5^{20}$ isn't divisible by any odd prime less than 19, so $\boxed{19}$ is the smallest odd prime factor.

20. The main idea is to use a partial fraction decomposition. Note that

$$\frac{1}{F_n}-\frac{1}{F_{n+2}}=\frac{F_{n+2}-F_n}{F_n\cdot F_{n+2}}=\frac{F_{n+1}}{F_n\cdot F_{n+2}}.$$

Therefore, dividing both sides by F_{n+1} gives

$$\frac{1}{F_nF_{n+2}}=\frac{1}{F_nF_{n+1}}-\frac{1}{F_{n+1}F_{n+2}}.$$

Now we can write

$$\begin{aligned}S&=\frac{1}{F_3F_5}+\frac{1}{F_4F_6}+\frac{1}{F_5F_7}+\cdots\\&=\frac{1}{F_3F_4}-\frac{1}{F_4F_5}+\frac{1}{F_4F_5}-\frac{1}{F_5F_6}+\frac{1}{F_5F_6}-\frac{1}{F_6F_7}+\cdots\\&=\frac{1}{F_3F_4}\\&=\frac{1}{6},\end{aligned}$$

since all the other terms cancel in a telescoping sum. Therefore $420S=\boxed{70}$.

21. Since Q is hard to deal with, we will construct the related but slightly different number

$$R=\sum_{i=0}^{\infty}\frac{F_i}{100^{i+1}}=0.0001010203050813213455904 4\ldots,$$

where we "squeeze" each Fibonacci number into two decimal spots, with carryover if necessary. This makes it so that our sum is much easier to evaluate. Note also that $R>Q$, or that $\frac{1}{R}<\frac{1}{Q}$.

To find this sum, we use a well-known telescoping technique. Defining $F_{-1}=1$:

$$\begin{aligned}100R-R&=F_0+\sum_{i=1}^{\infty}\frac{F_i}{100^i}-\sum_{i=1}^{\infty}\frac{F_{i-1}}{100^i}\\&=F_0+\sum_{i=1}^{\infty}\frac{F_{i-2}}{100^i}\\&=F_0+\frac{F_{-1}}{100}+\sum_{i=0}^{\infty}\frac{F_i}{100^{i+2}}\\&=\frac{1}{100}+\frac{1}{100}R.\end{aligned}$$

So

$$9900R=1+R\implies R=\frac{1}{9899}.$$

Now, note that the first 12 digits of R and Q are the same, so $R-Q\le 10^{-12}$. Thus,

$$\frac{1}{Q}-\frac{1}{R}=\frac{R-Q}{QR}<10^{-12}\cdot 10^4\cdot 10^4=10^{-4},$$

since $Q,R>10^{-4}$. Since $\frac{1}{R}$ is an integer, the greatest integer less than or equal to $\frac{1}{Q}$ is $\frac{1}{R}=\boxed{9899}$.

MOAA 2019 Gunga Bowl Set 8

22. The key is to think about this problem in terms of coordinates. Let the center of C_0 be the origin $(0,0,0,0,0)$, so its 32 vertices are $(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$. All these are a distance of

$$\sqrt{1^2+1^2+1^2+1^2+1^2} = \sqrt{5}$$

from the origin, so the radius of S_0 is $\sqrt{5}$. Therefore, the 32 vertices of C_1 are $(\pm\sqrt{5}, \pm\sqrt{5}, \pm\sqrt{5}, \pm\sqrt{5}, \pm\sqrt{5})$, and the radius of S_1 is 5. Continuing this way, we get that the 32 vertices of C_4 are $(\pm 25, \pm 25, \pm 25, \pm 25, \pm 25)$, so its side length is $\boxed{50}$.

23. Let's begin with a few lengths: $AD = DC = 10$, $BD = 5$, $AB = 5\sqrt{5}$. The area of $\triangle ABC$ is $\frac{1}{2} \cdot AD \cdot BC = 75$. Thus $BE \cdot AC = 150$, or $BE = \frac{15\sqrt{2}}{2}$. Having exhausted most of our length options, it makes sense to now turn to angles. Since $BFEC$ is cyclic, $\angle B$ and $\angle FEC$ are supplementary. However, $\angle AEF$ and $\angle FEC$ are also supplementary, so $\angle B = \angle AEF$. Thus, by AA similarity, $\triangle AEF \sim \triangle ABC$. Similarly, $\triangle BFD \sim \triangle BCA$ and $\triangle CDE \sim \triangle CAB$. Therefore we can calculate the side lengths of $\triangle DEF$ by similar triangles.

Before we do that, however, we can make a few more crucial observations about $\triangle DEF$ and $\triangle UVW$. note that since $\angle BFD = \angle C$ and $\angle AFE = \angle C$, we know that $\angle DFE = 180° - 2\angle C = 90°$! Thus $\triangle DFE$ is a right triangle. By a similar logic, we deduce that $\angle DEF = 180° - 2\angle B$ and $\angle EDF = 180° - 2\angle A$. Then, notice that $\triangle UVF$ is an isosceles triangle, since FU and FV are equal tangents to the incircle. Thus $\angle FUV = \angle C$. Similarly, $\angle EUW = \angle B$. This then means that $\angle VUW = 180° - \angle B - \angle C = \angle A$! By a similar logic, $\angle UVW = \angle B$ and $\angle UWV = \angle C$, so $\triangle UVW \sim \triangle ABC$. It suffices to find the length of UV, since then we can calculate the area of $\triangle UVW$ by using the appropriate scale factor. In order to do that, notice that since $\triangle DEF$ is a right triangle with a right angle at $\angle F$, the lengths FU and FV are both equal to the inradius of $\triangle DEF$. To see this, let H be the incenter of $\triangle DEF$, and notice that $\angle FUH = \angle FVH = \angle UFV = 90°$. Thus, since $FU = FV$, $FUHV$ is a square and the conclusion follows. Since we can find the side lengths of $\triangle DEF$, we can find the inradius of $\triangle DEF$ and therefore the length of UV in order to finish the problem.

Now that we know what we are doing, we can start the calculations. We first calculate $AE = \frac{3\sqrt{10}}{2}$ by the Pythagorean Theorem on $\triangle ABE$. By similar triangles,

$$\frac{AE}{EF} = \frac{AB}{BC} \implies EF = AE \cdot \frac{BC}{AB} = \frac{5\sqrt{2}}{2} \cdot \frac{15}{5\sqrt{5}} = \frac{3\sqrt{10}}{2}.$$

Similarly,

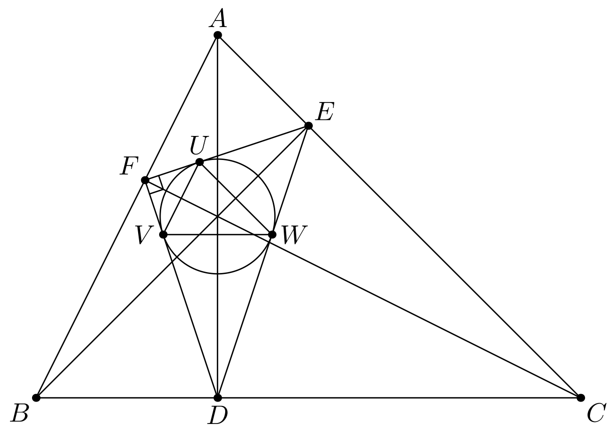
$$\frac{BD}{DF} = \frac{AB}{AC} \implies DF = BD \cdot \frac{AC}{AB} = 5 \cdot \frac{10\sqrt{2}}{5\sqrt{5}} = 2\sqrt{10}.$$

Finally,

$$\frac{CD}{DE} = \frac{AC}{AB} \implies DE = CD \cdot \frac{AB}{AC} = 10 \cdot \frac{5\sqrt{5}}{10\sqrt{2}} = \frac{5\sqrt{10}}{2}.$$

(We can perform a sanity check right now that, indeed, $DF^2 + EF^2 = DE^2$.) The area of $\triangle DEF$ is $DF \cdot EF = 15$. The inradius of a triangle with area K and semiperimeter s is $r = \frac{K}{s}$, so the inradius of $\triangle DEF$ is $\frac{15}{3\sqrt{10}} = \frac{\sqrt{10}}{2}$. Finally, we can use the fact that $UV = \sqrt{2}FU = \sqrt{2}r$ to see that $UV = \sqrt{5}$. The scale factor between $\triangle UVW$ and $\triangle ABC$ is thus $\frac{UV}{AB} = \frac{\sqrt{5}}{5\sqrt{5}} = \frac{1}{5}$. This means that the area of $\triangle UVW$ is

$$75 \cdot \left(\frac{1}{5}\right)^2 = \boxed{3}.$$



24. Suppose our polynomial is
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$
where a_i are selected from the set $S = \{0, 1, \dots, 8\}$. We need that
$$2019 = P(3) = a_0 + a_1 3 + a_2 3^2 + \cdots + a_n 3^n.$$
At this point, we can make a selection for the coefficients. We have a sum of $n+1$ terms: the first term can be chosen to be any element of S, while the second term can be chosen to be any element of $3S$, and so on. We can describe this by a generating function:
$$f(x) = (1 + x + x^2 + \cdots + x^8)(1 + x^3 + x^6 + \cdots + x^{24})\cdots$$
The coefficient of the x^{2019} term of f tells us how many ways we can select the a_i. However, we can recognize each term of the product as a geometric series and write
$$f(x) = \frac{x^9 - 1}{x - 1} \cdot \frac{x^{27} - 1}{x^3 - 1} \cdot \frac{x^{81} - 1}{x^9 - 1} \cdots,$$
which telescopes into
$$\frac{1}{(1-x)(1-x^3)} = (1 + x + x^2 + \cdots)(1 + x^3 + x^6 + \cdots).$$
Now note that for any selection of a multiple of three $3a$ less than or equal to 2019 in the second term, we can find the corresponding $2019 - 3a$ term in the first term. Therefore, the coefficient of the x^{2019} term is just the number of multiples of three at most 2019, which is $\boxed{674}$.

MOAA 2019 Gunga Bowl Set 9

25. We have that $A = 20192019^2 \cdot \pi$. The side length of a square with this area is
$$s = \sqrt{A} = 20192019 \cdot \sqrt{\pi} \approx 35789421.83.$$
Therefore, the answer is $\boxed{35789421}$. There are a few good ways to estimate this; one of the ways we found is to use a first-order Taylor series approximation of $f(x) = \sqrt{x}$ centered at $x = 3.24$ to approximate the value of $\sqrt{\pi}$:
$$\begin{aligned}\sqrt{x} &\approx f(3.24) + f'(3.24)(x - 3.24) + \cdots \\ &= \sqrt{3.24} + \frac{1}{2 \cdot \sqrt{3.24}}(x - 3.24) + \cdots \\ &= 1.8 + \frac{1}{3.6}(x - 3.24) + \cdots\end{aligned}$$

for values of x around 3.24. Dropping the higher order terms and approximating $\pi \approx 3.14$, we find that $\sqrt{\pi} \approx 1.8 - \frac{1}{36}$. We can choose to approximate 20192019 as 20190000 to make the calculations easier (this is off by about 0.01%). Then

$$\begin{aligned} 20190000\left(1.8 - \frac{1}{36}\right) &= 2019\left(18000 - \frac{10000}{36}\right) \\ &= 2019\left(18000 - \frac{2500}{9}\right) \\ &\approx 2019\left(18000 - 278\right) \\ &= 2019(17722) \\ &= 35780718. \end{aligned}$$

This is an extremely good approximation of the actual answer (within 0.025%), and earns a solid 25 points. Note that other close-ish answers, such as approximating $\sqrt{\pi}$ as 1.77, only earns 4 points.

26. This problem asks for the domination number of a 50×50 grid graph. It has been proven that for $m, n \geq 16$, the domination number of a $m \times n$ grid graph is exactly $\left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor - 4$. Plugging in $m = n = 50$ gives $\boxed{536}$.

27. It has been proven that the optimal packing density is $d = \frac{\pi^4}{2^4 \cdot 4!} \approx 25.37\%$, achieved by the E8 lattice. (This is essentially the most regular spherical packing in any dimension. Think hexagonal circle packing in 2 dimensions or hexagonally-layered sphere packing in 3 dimensions.) The desired answer is then $\lfloor 10^8 \cdot d \rfloor = \boxed{25366950}$.