MOAA 2019 Speed Round Solutions

1. Keeping in mind order of operations, we first perform the operation in parenthesis. Since $2 - 7 = -5$, we get
$$20 \times 19 + 20 \div (-5).$$
Now we perform the multiplication and division operations, to get
$$380 + (-4).$$
Finally, we add these two numbers to get
$$380 + (-4) = 380 - 4 = \boxed{376}.$$

2. The probability the second spinner matches the first is $\frac{1}{4}$, and the probability the third spinner matches the first is $\frac{1}{5}$. Multiplying these gives $p = \frac{1}{20}$, so the answer is $\frac{1}{p} = \boxed{20}$.

3. There are $\binom{8}{5} = 56$ total ways to seat the eight children. When two boys sit on the end, there are $\binom{6}{3} = 20$ ways to seat the six remaining children. Thus the probability that two boys sit on the end when the children are seated randomly is $\frac{20}{56} = \frac{5}{14}$, and our answer is $\boxed{19}$.

4. Suppose that, before his 10 home run streak, Jaron hit x home runs in y at-bats. Then by the conditions given, $\frac{x}{y} = 0.3$ and $\frac{x+10}{y+10} = 0.31$. Cross multiplying yields $x = 0.3y$ and $x + 10 = 0.31y + 3.1$, and substituting the first equation into the second gives
$$\begin{aligned} 0.3y + 10 &= 0.31y + 3.1 \\ 6.9 &= 0.01y \\ y &= 690. \end{aligned}$$
Thus $x = 0.3y = 207$, and after hitting 10 home runs in a row, Jaron has now hit $\boxed{217}$ home runs.

5. Note that we may telescope the sum by writing $\frac{1}{n(n+3)} = \frac{1}{3}(\frac{1}{n} - \frac{1}{n+3})$. (In general, we can show that $\frac{1}{n(n+a)} = \frac{1}{a}(\frac{1}{n} - \frac{1}{n+a})$.) Thus, the sum becomes
$$\begin{aligned} \frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \cdots + \frac{1}{97 \cdot 100} &= \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4} + \frac{1}{4} - \frac{1}{7} + \cdots + \frac{1}{97} - \frac{1}{100}\right) \\ &= \frac{1}{3}\left(\frac{1}{1} - \frac{1}{100}\right) \\ &= \frac{33}{100}. \end{aligned}$$
The answer is thus $\boxed{133}$.

6. Let I be the area of the intersection between the square and triangle. Then, notice that
$$|M - N| = |(M + I) - (N + I)|.$$
However, $M + I$ is the area of the square and $N + I$ is the area of the triangle, so we can just find the difference between these two areas.

The unit square clearly has area 1. For the triangle, note that $OA = \frac{\sqrt{2}}{2}$. If G is the midpoint of OE, then OAG is a $30-60-90$ triangle, so $OG = \frac{\sqrt{6}}{4}$ and $OE = \frac{\sqrt{6}}{2}$. Now since the area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$, the area of OEF is

$$N + I = \frac{\sqrt{3}}{4} \cdot \left(\frac{\sqrt{6}}{2}\right)^2 = \frac{3\sqrt{3}}{8}.$$

Therefore the difference between the two areas is

$$1 - \frac{\sqrt{27}}{8} = \frac{1}{8}(8 - \sqrt{27}),$$

so the answer is $8 + 27 = \boxed{35}$.

7. Let's do casework on the first digit. If the first digit is $1 \pmod 3$, then the second digit must be equivalent to $2 \pmod 3$ in order for the sum of the two digits to be $0 \pmod 3$. Then the third digit must be $1 \pmod 3$, and so on, so that the digit alternate between $1 \pmod 3$ and $2 \pmod 3$. Similarly, if the first digit is equivalent to $2 \pmod 3$, then the digits of the number must alternate between $2 \pmod 3$ and $1 \pmod 3$. Since there are three digits that are $1 \pmod 3$ (1, 4, 7) and three digits that are $2 \pmod 3$ (2, 5, 8), there are 3 ways to pick each digit in each of these cases, meaning that there are a total of $2 \cdot 3^7 = 4374$ numbers starting with a digit that is either $1 \pmod 3$ or $2 \pmod 3$.

   If the number starts with a digit that is $0 \pmod 3$, then each digit must be equivalent to $0 \pmod 3$ in order for adjacent digits to sum to a multiple of 3. There are four digits that are $0 \pmod 3$ (0, 3, 6, 9), but a number cannot start with 0. Thus there are three choices for the leading digit, and four choices for each of the other digits. There are $3 \cdot 4^6 = 12288$ numbers in this case.

   Adding our cases, the total number of 7-digit numbers with adjacent digits summing to a multiple of 3 is $4374 + 12288 = \boxed{16662}$.

8. Suppose first that $x = p^e$ is a prime power. Then $x^x = (p^e)^{p^e} = p^{ep^e}$. This number has $ep^e + 1 = 703$ factors, meaning that $ep^e = 702$. However, $702 = 2 \cdot 3^3 \cdot 13$, so either $p = 13$ or e is a multiple of 13. If $p = 13$ this forces $e = 1$, which clearly does not satisfy the equation. Alternatively, if e is a multiple of 13, then

   $$ep^e > 2^{13} = 8192 > 702,$$

   so there are no solutions here either. Thus x has at least two distinct prime factors.

   Since the number of factors of $n = p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ is $(e_1+1)(e_2+1)\cdots(e_k+1)$, each prime factor will contribute a factor to the final product, 703. Thus we factor $703 = 19 \cdot 37$. Since 703 has exactly two prime factors, x must have exactly two prime factors. Suppose that $x = p^a q^b$ for distinct primes p, q and positive integers a, b. Then $x^x = p^{ax}q^{bx}$. WLOG suppose that $ax + 1 = 19$ and $bx + 1 = 37$, or that $ax = 18$ and $bx = 36$. This forces x to be a factor of 18 with two distinct prime factors. The only possible candidates are 6 and 18. Testing 6, we find that $6^6 = 2^6 \cdot 3^6$ has 49 factors. Testing 18, we find that $18^{18} = 2^{18} \cdot 3^{36}$ has 703 factors. Thus, our answer is $\boxed{18}$.

9. Let's first try some smaller cases for the numbers in place of 2019.

x	2^x	5^x	total number of digits
1	2	5	2
2	4	25	3
3	8	125	4
4	16	625	5
5	32	3125	6

Hmm... there seems to be a pattern developing. The total number of digits in 2^x and 5^x seems to be $x+1$.

Multiplying 2^{2019} and 5^{2019} gives 10^{2019}, which has 2020 digits. Now we have a pretty clear reason why the total number of digits should be 2020. Write

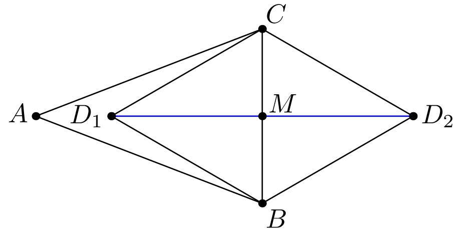
$$2^{2019} = a \cdot 10^b \text{ and } 5^{2019} = c \cdot 10^d$$

in scientific notation, with a and c between 1 and 10. 2^{2019} has $b+1$ digits and 5^{2019} has $d+1$ digits, so now we just need to find $b+d+2$. However,

$$10^{2019} = 2^{2019} \cdot 5^{2019} = ac \cdot 10^{b+d}.$$

Noting that $1 < a, c < 10$, in order the right hand side to be a power of 10, we must have $ac = 10$, so $10^{2019} = 10 \cdot 10^{b+d}$, so $b + d = 2018$. Our answer is in fact $b + d + 2 = \boxed{2020}$.

10. Consider the following diagram.



We can see that ω is the circle centered at M, the midpoint of BC, passing through D_1 and D_2, two points where D_1BC and D_2BC are equilateral triangles. In the above figure, the circle is coming out of the page, and the below figure shows the same arrangement rotated 90 degrees about the axis AM.

Since $MC = 5$ and $AC = 13$, and ACM is a right triangle, we know that $AM = 12$. The radius of ω is

$$D_1M = \sqrt{3} \cdot CM = 5\sqrt{3}.$$

Now we know that

$$AE = \sqrt{AM^2 - EM^2} = \sqrt{144 - 75} = \sqrt{69}.$$

Now if X is the intersection of AM and AE, notice that since $\angle AXE = \angle AEM = 90°$, by AA similarity,

$$\triangle AXE \sim \triangle AEM$$

Therefore

$$\frac{XE}{AE} = \frac{EM}{AM} \implies XE = \frac{5\sqrt{3} \cdot \sqrt{69}}{12} = \frac{5\sqrt{23}}{4}.$$

Then $EF = 2XE$, so

$$EF^2 = \left(\frac{5\sqrt{23}}{2}\right)^2 = \frac{575}{4},$$

giving us an answer of $575 + 4 = \boxed{579}$.