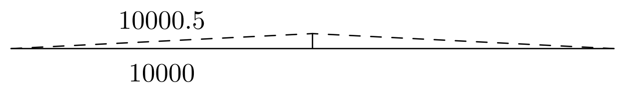
MOAA 2019 Team Round Solutions

1. The distance from an end of the bridge to the center of the bridge is 10000 meters, and the distance from an end of the bridge to the top of the pole is 10000.5 meters. Thus, the length of the pole is

$$\begin{aligned}\sqrt{10000.5^2 - 10000^2} &= \sqrt{(10000.5 + 10000)(10000.5 - 10000)} \\ &= \sqrt{20000.5 \cdot 0.5} \\ &= \sqrt{10000.25}.\end{aligned}$$

The closest integer to $\sqrt{10000.25}$ is $\boxed{100}$.



Note: Figure not drawn to scale.

2. Let r and s be the two distinct roots of the given quadratic. The hypotenuse of the right triangle with legs of length r and s is $\sqrt{r^2 + s^2}$. However, by Vieta's Formulas, we have $r + s = 36$ and $rs = 70$. Now simply note that

$$\sqrt{r^2 + s^2} = \sqrt{(r + s)^2 - 2rs} = \sqrt{36^2 - 140} = \boxed{34}.$$

3. Let r and s be the roots to $x^2 - ax + b$. By Vieta's formulas (or by just setting $(x-r)(x-s) = x^2 - ax + b$), we have $r + s = a$ and $rs = b$. Let's focus on the first equation. Since a is positive, r and s can't both be negative. Furthermore, if only one of r and s is negative, then b would be negative, and if either of r and s is 0, then so too is b, which is not a valid value. Thus it follows that r and s must both be positive integers. For each pair of positive integers satisfying $r + s = a$, there is a unique positive value of b such that $rs = b$. Therefore, for each value of a, we only need to calculate the values of r and s which give distinct values of b.

One problem remains: which values of r and s give distinct values of b? Since the quadratic is monic (the leading coefficient is 1), the values of r and s uniquely characterize the quadratic. Therefore, as long as r and s are different, then the values of a and b will be different.

When $a = 1$, there are no solutions. When $a = 2$, there is one solution, $(r, s) = (1, 1)$. When $a = 3$, there is also one solution, $(r, s) = (2, 1)$. (Note that it might be tempting to say that $(r, s) = (1, 2)$ is another solution, but it is actually the same solution since the set of roots is the same.) Thus, for all even values of $a = 2k$, r can assume any value from 1 to k, and s will assume the corresponding value from $2k - 1$ to k. For all odd values of $a = 2k + 1$, r can assume any value from 1 to k, and s will assume the corresponding value from $2k$ to $k + 1$. Our answer is therefore

$$1 + 1 + 2 + 2 + \cdots + 24 + 24 + 25 = 24 \cdot 25 + 25 = \boxed{625}.$$

4. Consider the basses first. Since each group must have at least one bass, we first add one bass to each group and have 2 left to place. We can place these two remaining basses in $3 + \binom{3}{2} = 6$ ways. This is

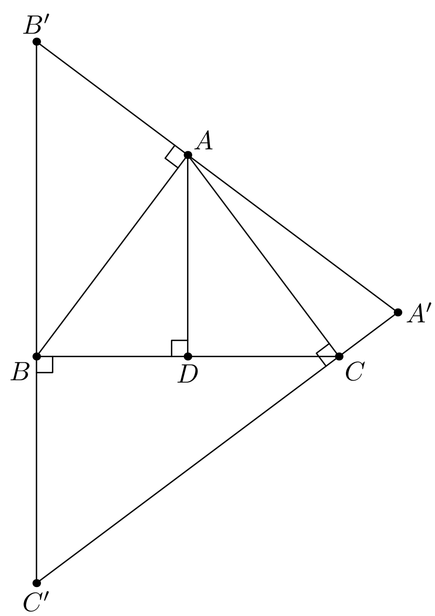
because we can either place both basses in the same group (3 ways), or we can choose two of the three groups to receive one bass ($\binom{3}{2}$ ways).

Now consider the cellos. Since there must be more cellos than basses, we first divide 5 cellos in the same way as the basses, and then add an extra cello to each group. After dividing these 8 cellos, we have one more cello in each group than basses. Now, we can divide up the remaining 2 cellos in 6 ways, exactly the same as we did with the basses. We use the same process for violas and violins and find that Brandon has $6 \cdot 6 \cdot 6 \cdot 6 = \boxed{1296}$ ways to divide his orchestra.

5. Let $A' = \ell_C \cap \ell_A$, $B' = \ell_A \cap \ell_B$, and $C' = \ell_B \cap \ell_C$. The first observation is that $\triangle ABC \sim \triangle A'B'C'$. Indeed, note that both $\angle AB'B$ and $\angle ABC$ are complementary to $\angle ABB'$, and that both $\angle BC'C$ and $\angle ACB$ are complementary to $\angle BCC'$. Thus $\triangle ABC \sim \triangle A'B'C'$ by AA similarity.

   Since the perimeter of $\triangle ABC$ is 32, it suffices to find the scale factor between the two triangles. Observe first that $AD = 8$ by the Pythagorean Theorem. Note that since $\angle AB'B = \angle ABD$ and $\angle BAB' = \angle ADB = 90°$, $\triangle ADB \sim \triangle B'AB$. Thus $BB' = AB \cdot \frac{AB}{AD} = 10 \cdot \frac{10}{8} = \frac{25}{2}$.

   Similarly, since $\angle BC'C = \angle ACD$ and $\angle CBC' = \angle ADC = 90°$, $\triangle ADC \sim \triangle C'BC$. Once again, $BC' = CD \cdot \frac{BC}{AD} = 6 \cdot \frac{12}{8} = 9$. Therefore, $B'C' = B'B + BC' = \frac{43}{2}$. The corresponding length in $\triangle ABC$, BC, has length 12. Thus the scale factor between $\triangle ABC$ and $\triangle A'B'C'$ is $\frac{43}{24}$, and the perimeter of $\triangle A'B'C'$ is $32 \cdot \frac{43}{24} = \frac{172}{3}$. The answer is $\boxed{175}$.



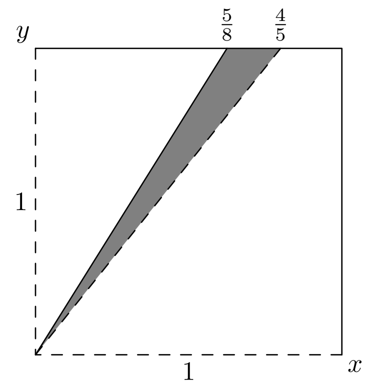
6. First, consider $g(x, y) = \frac{5x}{2y} + \frac{5y}{2x}$. By the AM-GM inequality,

$$\frac{g(x,y)}{2} \geq \sqrt{\frac{5x}{2y} \cdot \frac{5y}{2x}} = \frac{5}{2} \implies g(x,y) \geq 5.$$

   Now note that $f(x, y) > g(x, y) - 1 \geq 4$, since the floor function decreases g by less than 1, while the ceiling function only increases it. However, since $f(x, y)$ is an integer, we know that $f(x, y) \geq 5$. Since we want $f(x, y) < 6$, it suffices to look at when $f(x, y) = 5$. Now let's do casework on the values of the floor and ceiling function. Note first that since $x, y > 0$, we know that $\lfloor \frac{5x}{2y} \rfloor \geq 0$ and $\lceil \frac{5y}{2x} \rceil \geq 1$.

   - Case 1: $\lfloor \frac{5x}{2y} \rfloor = 0$ and $\lceil \frac{5y}{2x} \rceil = 5$. Then we need $0 \leq \frac{5x}{2y} < 1$ and $4 < \frac{5y}{2x} \leq 5$, or $y > \frac{5}{2}x$ and $\frac{8}{5}x < y \leq 2x$. This is clearly impossible, since it implies that $\frac{5}{2}x < y \leq 2x$, which is false.

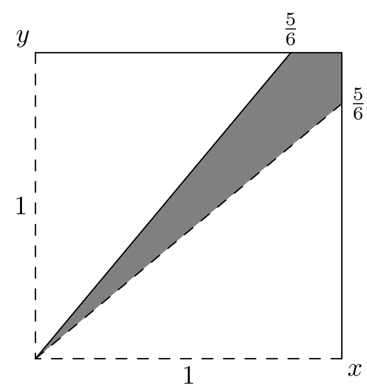
- Case 2: $\lfloor \frac{5x}{2y} \rfloor = 1$ and $\lceil \frac{5y}{2x} \rceil = 4$. Then we need $1 \le \frac{5x}{2y} < 2$ and $3 < \frac{5y}{2x} \le 4$, or $\frac{2}{5}y \le x < \frac{4}{5}y$ and $\frac{6}{5}x < y \le \frac{8}{5}x$. The limiting upper bound for y is $y \le \frac{8}{5}x$, and the limiting lower bound for y is $y > \frac{5}{4}x$. When these two conditions are satisfied, we get this case.



The shaded triangle shows the space of solutions to this case in the space of all (x, y). The corresponding probability for this case is the area of the shaded space divided by the area of the total space, or

$$\frac{1/2 \cdot (4/5 - 5/8)}{1} = \frac{7}{80}.$$

- Case 3: $\lfloor \frac{5x}{2y} \rfloor = 2$ and $\lceil \frac{5y}{2x} \rceil = 3$. Then we need $2 \le \frac{5x}{2y} < 3$ and $2 < \frac{5y}{2x} \le 3$, or $\frac{4}{5}y \le x < \frac{6}{5}y$ and $\frac{4}{5}x < y \le \frac{6}{5}x$. The limiting upper bound for y is $y \le \frac{6}{5}x$, and the limiting lower bound for y is $y > \frac{5}{6}x$. When these two conditions are satisfied, we get this case.

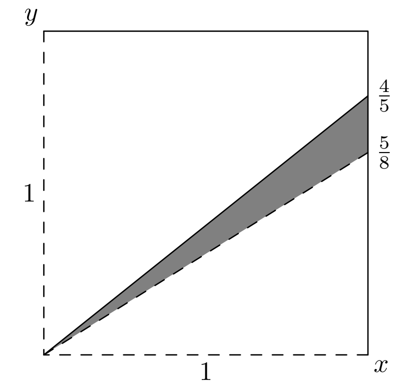


The shaded space shows the space of solutions to this case in the space of all (x, y). The corresponding probability for this case is the area of the shaded space divided by the area of the total space, or

$$2 \cdot \frac{1/2 \cdot (1 - 5/6)}{1} = \frac{1}{6}.$$

Note that there is no overlap with the previous section, since $\frac{5}{6} > \frac{4}{5}$.

- Case 4: $\lfloor \frac{5x}{2y} \rfloor = 3$ and $\lceil \frac{5y}{2x} \rceil = 2$. Then we need $3 \le \frac{5x}{2y} < 4$ and $1 < \frac{5y}{2x} \le 2$, or $\frac{6}{5}y \le x < \frac{8}{5}y$ and $\frac{2}{5}x < y \le \frac{4}{5}x$. The limiting upper bound for y is $y \le \frac{4}{5}x$, and the limiting lower bound for y is $y > \frac{5}{8}x$. When these two conditions are satisfied, we get this case.



The shaded triangle shows the space of solutions to this case in the space of all (x, y). The corresponding probability for this case is the area of the shaded region divided by the area of the total space, or

$$\frac{1/2 \cdot (4/5 - 5/8)}{1} = \frac{7}{80}.$$

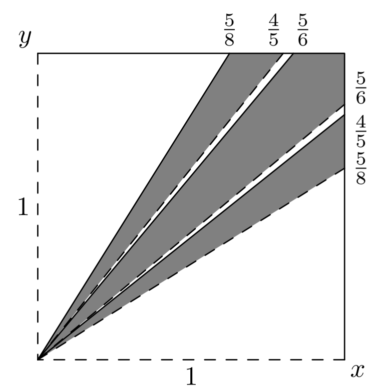
Note that, for the same reasons, there is no overlap with previous sections.

- Case 5: $\lfloor \frac{5x}{2y} \rfloor = 4$ and $\lceil \frac{5y}{2x} \rceil = 1$. Then we need $4 \leq \frac{5x}{2y} < 5$ and $0 < \frac{5y}{2x} \leq 1$, or $\frac{8}{5}y \leq x < 2y$ and $y \leq \frac{2}{5}x$. This is clearly impossible, since it implies that $2y > x \geq \frac{5}{2}y$, which is false.

Since the regions we found were pairwise disjoint, the probability of (x, y) satisfying the problem condition is equal to the sum of the individual probabilities of each of the cases. Thus the probability that $f(x, y) < 6$ is

$$\frac{7}{80} + \frac{1}{6} + \frac{7}{80} = \frac{41}{120},$$

and the answer is therefore $\boxed{161}$. Shown below is the complete solution space.



7. The key claim is that G is the centroid of $\triangle AHA'$, where H is the orthocenter of ABC. First, we claim that M, the midpoint of BC, is also the midpoint of HA'. In fact, we can notice that

$$\angle A'BH = \angle ABA' - \angle ABH = 90^\circ - \angle ABH = \angle BAC = 180^\circ - \angle BHC,$$

where the second equality holds because AA' is a diameter and the third equality holds because $BH \perp AC$. Therefore $BA' \parallel HC$ and similarly $CA' \parallel HB$, so $BHCA'$ is a parallelogram. Then its diagonals must bisect each other so M is indeed the midpoint of HA'.

One defining characteristic of the centroid is that it is the trisection point on median AM that is closer to M. Since AM is also a median of AHA', and G is a trisection point closer to M, we have confirmed that G is also the centroid of AHA'. Therefore, the area of AGA' is 1/3 that of AHA'.

To compute the area of AHA', let H' be the reflection of H over D, the foot of the altitude from A to BC. We claim that H' lies on ω. This follows from quick angle chasing:

$$\angle BH'C = \angle BHC = 180^\circ - \angle BAC.$$

Now, the area of AHA' is

$$[AHA'] = \frac{1}{2} \cdot AH \cdot H'A'$$

since $\angle AH'A' = 90^\circ$. Now we can compute some lengths. By Heron's Formula, the area of ABC is

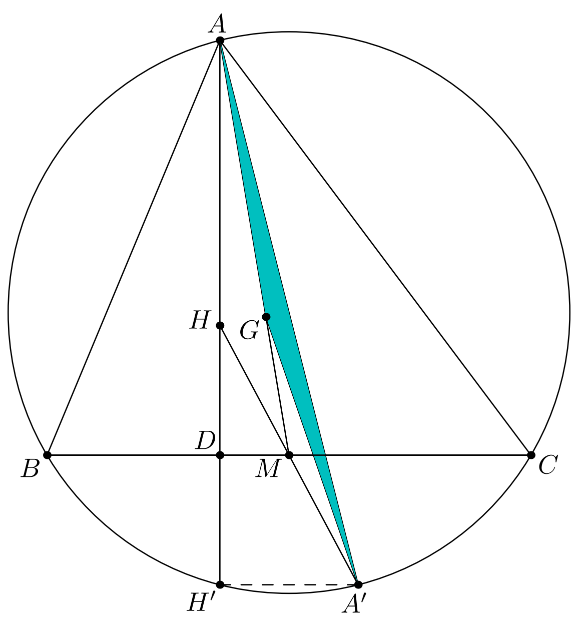
$$\sqrt{21 \cdot 6 \cdot 7 \cdot 8} = 84,$$

so $AD = 12$. Then $BD = 5$ and $CD = 9$. By Power of a Point at D, we know that

$$AD \cdot DH' = BD \cdot CD \implies 12 \cdot DH' = 5 \cdot 9 \implies DH' = \frac{15}{4}.$$

Since H' is the reflection of H over D, we also know that $DH = \frac{15}{4}$, so $AH = AD - HD = 12 - \frac{15}{4} = \frac{33}{4}$.

To compute $H'A'$, since D is the midpoint of HH' and M is the midpoint of HA', DM is a midline of triangle $HH'A'$ and is half the length of $H'A'$. However, since $DC = 9$ and $MC = \frac{1}{2}BC = 7$, $DM = 2$ and therefore $H'A' = 4$. The area of AHA' is $\frac{1}{2} \cdot \frac{33}{4} \cdot 4 = \frac{33}{2}$, so the area of AGA' is 1/3 of that, or $\frac{11}{2}$. Then, $100n + m = \boxed{211}$.



8. We multiply both sides of the equation by

$$\frac{(\sqrt{2})^5 - 1}{\sqrt{2} - 1} = 7 + 3\sqrt{2}.$$

Then,

$$
\begin{aligned}
m &= \left[\frac{(\sqrt{2})^5-1}{\sqrt{2}-1}\cdot\frac{(\sqrt{2})^5+1}{\sqrt{2}+1}\right]\cdot\frac{2^5+1}{2+1}\cdot\frac{4^5+1}{4+1}\cdot\frac{16^5+1}{16+1}\\
&= \left[\frac{2^5-1}{2-1}\cdot\frac{2^5+1}{2+1}\right]\cdot\frac{4^5+1}{4+1}\cdot\frac{16^5+1}{16+1}\\
&\vdots\\
&= \frac{256^5-1}{256-1}
\end{aligned}
$$

by collapsing the terms in square brackets using difference of squares. The resulting expression is a geometric series, which expands to

$$m = 1 + 256 + 256^2 + 256^3 + 256^4 = 1 + 2^8 + 2^{16} + 2^{24} + 2^{32}.$$

In binary, this is

$$100000001000000010000000100000001_2,$$

a 33-digit number with 5 1's. There are a total of $\boxed{28}$ digits that are 0.

9. To begin, note that this is equivalent to summing only the a's and then multiplying by 3 at the end. This is because a, b, c are symmetric and so each variable will have the same sum.

Now, for each factor of 720 that a assumes, b and c must be other factors of 720 that multiply to $\frac{720}{a}$. There are exactly $d(\frac{720}{a})$ such pairs (b, c), where $d(n)$ is the number of divisors of n. Therefore, each factor a of 720 contributes $a \cdot d(\frac{720}{a})$ to the overall sum, which is $\sum_{a|720} a \cdot d(\frac{720}{a})$. (The notation just means that we are summing terms of the form $a \cdot d(\frac{720}{a})$ for all positive integers a that divide 720.)

At this point, there are multiple ways to finish. One obvious way is to brute-force add over all factors of 720. We instead take a more strategic approach by proving that $f(n) = \sum_{a|n} a \cdot d(\frac{n}{a})$ is multiplicative (i.e. $f(m)f(n) = f(mn)$ when m and n are relatively prime positive integers). Thus, we need only to compute f on the prime powers in order to complete the problem.

To prove that f is multiplicative, we show the following more general fact:

Lemma. If f and g are multiplicative functions, then so is the function $f * g$ defined by

$$(f * g)(n) = \sum_{a|n} f(a) g\left(\frac{n}{a}\right).$$

Proof. For convenience define $h = f * g$. Consider relatively prime positive integers p and q, and consider

$$h(pq) = \sum_{a|pq} f(a) g\left(\frac{pq}{a}\right).$$

Now, for each a diving pq, write $a = xy$, where $x \mid p$ and $y \mid q$. This decomposition of a is unique, since

p and q share no prime factors. Then we can write

$$\begin{aligned} h(pq) &= \sum_{xy|pq} f(xy)g\left(\frac{pq}{xy}\right) \\ &= \sum_{xy|pq} f(x)g\left(\frac{p}{x}\right) \cdot f(y)g\left(\frac{q}{y}\right) \\ &= \sum_{x|p} f(x)g\left(\frac{p}{x}\right) \cdot \sum_{y|q} f(y)g\left(\frac{q}{y}\right) \\ &= h(p)h(q), \end{aligned}$$

where we have used the fact that f and g are multiplicative. This proves the lemma. □

Now consider n and $d(n)$. Clearly n is multiplicative. To show that d is multiplicative, simply note that for $m = p_1^{e_1} p_2^{e_2} \dots p_i^{e_i}$ and $n = q_1^{f_1} q_2^{f_2} \dots q_j^{f_j}$ relatively prime, we have

$$d(mn) = (e_1 + 1)(e_2 + 1) \dots (e_i + 1)(f_1 + 1)(f_2 + 1) \dots (f_j + 1) = d(m)d(n).$$

Thus, $f(n) = \sum_{a|n} a \cdot d(\frac{n}{a})$ is also multiplicative by our lemma.

It remains to calculate f for prime powers. Noting that $720 = 2^4 \cdot 3^2 \cdot 5$, we need only to compute $f(p)$, $f(p^2)$, and $f(p^4)$. Note that

$$f(p) = \sum_{a|p} a \cdot d\left(\frac{p}{a}\right) = 1 \cdot 2 + p \cdot 1 = p + 2,$$

$$f(p^2) = \sum_{a|p^2} a \cdot d\left(\frac{p^2}{a}\right) = 1 \cdot 3 + p \cdot 2 + p^2 \cdot 1 = p^2 + 2p + 3,$$

$$f(p^4) = \sum_{a|p^4} a \cdot d\left(\frac{p^4}{a}\right) = 1 \cdot 5 + p \cdot 4 + p^2 \cdot 3 + p^3 \cdot 2 + p^4 \cdot 1 = p^4 + 2p^3 + 3p^2 + 4p + 5.$$

Finally, we can compute the answer of

$$3f(720) = 3f(2^4)f(3^2)f(5) = 3(2^4 + 2 \cdot 2^3 + 3 \cdot 2^2 + 4 \cdot 2 + 5)(3^2 + 2 \cdot 3 + 3)(5 + 2) = \boxed{21546}.$$

Remark. Some experienced contestants may recognize $f * g$ as the *Dirichlet convolution* of f and g, where it is well-known that $f * g$ is multiplicative when f and g are multiplicative, by the proof above.

10. All four-digit palindromes can be expressed in the form $\overline{abba}$ for digits a and b, where a is nonzero. Note that given two four-digit palindromes $\overline{abba}$ and $\overline{cddc}$, the comparison between a and c is the deciding factor in determining which number is greater. Since we are only considering their absolute difference, WLOG we can assume that $a > c$ or $a = c$. Let's consider $a > c$ first. Note that since the case $a < c$ is identical, we can perform the calculations in this case and then multiply by 2. Also note that

$$|\overline{abba} - \overline{cddc}| + |\overline{adda} - \overline{cbbc}| = \overline{a00a} + \overline{bb0} - \overline{c00c} - \overline{dd0} + \overline{a00a} + \overline{dd0} - \overline{c00c} - \overline{bb0} = 2\left(\overline{a00a} - \overline{c00c}\right),$$

which means that when $a > c$ (equivalently $a < c$), the middle two digits will always cancel out in the total sum by interchanging them between the two numbers. Thus, we need only to sum the outer digits in this case. Observe that there are an equal number of palindromes that begin with any given digit (specifically, 10 for each digit). Therefore, we just need to find the sum of all possible differences between a and c, then multiply by $10 \cdot 10 = 100$ for the number of ways we can then choose b and d. Noting that $a > c > 0$, we find all possible differences between a and c. When $a = 9$, c can be anything from 1 to 8, for a total sum of the $a - c$ to equal 36. In a similar fashion, we deduce that for each a, the total sum for that case equals $\binom{a}{2}$. The total sum is then

$$\binom{2}{2} + \binom{3}{2} + \cdots + \binom{9}{2} = \binom{10}{3} = 120.$$

So in the case that $a > c$, the sum of the absolute differences of the palindromes equals

$$120 \cdot 100 \cdot 1001 = 12012000.$$

The 1001 comes from the fact that each a contributes $1001a$ to the sum, since it appears in the thousands and ones digits. Then, since the case $a < c$ is symmetric, the total sum for $a \neq c$ is $2 \cdot 12012000 = 24024000$.

We now consider $a = c$. In this case, we may ignore the outer digits a and c and focus only on b and d, remembering to multiply by 9 at the end for the number of choices of $a = c$. The calculation is similar: we assume $b > d \geq 0$, and then calculate the sum of all the absolute differences between b and d. At the end, we multiply by 2 to account for the symmetric case $b < d$. The case $b = d$ doesn't matter, since it just gives us 0. The difference in this case, however, is that b and d can reach 0. This means that for each b, the sum of all the absolute differences for the possible values of d is equal to $\binom{b+1}{2}$. Thus, the total sum of the absolute differences $|b - d|$ when $b > d \geq 0$ is

$$\binom{2}{2} + \binom{3}{2} + \cdots + \binom{10}{2} = \binom{11}{3} = 165.$$

Thus the total sum when $b > d \geq 0$ is

$$165 \cdot 9 \cdot 110 = 163350,$$

where the 110 comes from the fact that b and d occupy the hundreds and tens places. Finally, since the case $b < d$ is symmetric, the total sum when $a = c$ is $2 \cdot 163350 = 326700$.

Adding together our two cases, the total sum of all the absolute differences of four-digit palindromes is

$$24024000 + 326700 = 24350700.$$

Since there are $9 \cdot 10 = 90$ four-digit palindromes, there are $90 \cdot 90 = 8100$ absolute differences summed, and so the average value of the absolute difference is

$$\frac{24350700}{8100} = \frac{81169}{27}.$$

The final answer is thus $\boxed{81196}$.