MOAA 2020: General Round

October 10th, 2020, 11:15AM to 12:00PM Eastern Time

Rules

- No mathematical texts, notes, or online resources of any kind are permitted. Rely on your brain and those of your teammates!
- Compasses, protractors, rulers, straightedges, graph paper, blank scratch paper, and writing implements are generally permitted, so long as they are not designed to give an unfair advantage.
- No computational aids (including but not limited to calculators, phones, calculator watches, and computer programs) are permitted on any portion of the MOAA.
- Individuals may only help or receive help from members of their team — consulting any other individual is grounds for disqualification.

How to Compete

- Participants are recommended to use a video calling software which supports screen sharing such as Zoom to communicate with members of their team. They may also use collaborative whiteboard software such as Miro to share diagrams.
- After completing the test, the team captain should input answers to the problems in the submission portal for their registered team account.

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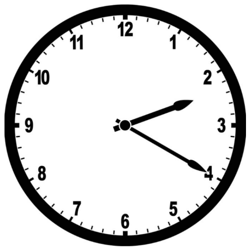
General Round Problems

The General Round consists of 20 problems, ordered in approximately increasing difficulty, to be solved in 45 minutes. All answers are nonnegative integers no larger than 1,000,000.

G1. What is $20 \times 20 - 19 \times 19$?

G2. Andover has a total of 1440 students and teachers as well as a $1 : 5$ teacher-to-student ratio (for every teacher, there are exactly 5 students). In addition, every student is either a boarding student or a day student, and 70% of the students are boarding students. How many day students does Andover have?

G3. The time is 2:20. If the acute angle between the hour hand and the minute hand of the clock measures x degrees, find x.



G4. Point P is located on segment AC of square $ABCD$ with side length 10 such that $AP > CP$. If the area of quadrilateral $ABPD$ is 70, what is the area of $\triangle PBD$?

G5. Andrew always sweetens his tea with sugar, and he likes a $1 : 7$ sugar-to-unsweetened tea ratio. One day, he makes a 100ml cup of unsweetened tea but realizes that he has run out of sugar. Andrew decides to borrow his sister's jug of pre-made SUPERSWEET tea, which has a $1 : 2$ sugar-to-unsweetened tea ratio. How much SUPERSWEET tea, in ml, does Andrew need to add to his unsweetened tea so that the resulting tea is his desired sweetness?

G6. Jeremy the architect has built a railroad track across the equator of his spherical home planet which has a radius of exactly 2020 meters. He wants to raise the entire track 6 meters off the ground, everywhere around the planet. In order to do this, he must buy more track, which comes from his supplier in bundles of 2 meters. What is the minimum number of bundles he must purchase? Assume the railroad track was originally built on the ground.

G7. Mr. DoBa writes the numbers $1, 2, 3, \ldots, 20$ on the board. Will then walks up to the board, chooses two of the numbers, and erases them from the board. Mr. DoBa remarks that the average of the remaining 18 numbers is exactly 11. What is the maximum possible value of the larger of the two numbers that Will erased?

G8. Nathan is thinking of a number. His number happens to be the smallest positive integer such that if Nathan doubles his number, the result is a perfect square, and if Nathan triples his number, the result is a perfect cube. What is Nathan's number?

G9. Let S be the set of positive integers whose digits are in strictly increasing order when read from left to right. For example, 1, 24, and 369 are all elements of S, while 20 and 667 are not. If the elements of S are written in increasing order, what is the 100th number written?

G10. Find the largest prime factor of the expression

$$2^{20} + 2^{16} + 2^{12} + 2^8 + 2^4 + 1.$$

G11. Christina writes down all the numbers from 1 to 2020, inclusive, on a whiteboard. What is the sum of all the digits that she wrote down?

G12. Triangle ABC has side lengths $AB = AC = 10$ and $BC = 16$. Let M and N be the midpoints of segments BC and CA, respectively. There exists a point $P \neq A$ on segment AM such that $2PN = PC$. What is the area of $\triangle PBC$?

G13. Consider the polynomial

$$P(x) = x^4 + 3x^3 + 5x^2 + 7x + 9.$$

Let its four roots be a, b, c, d. Evaluate the expression

$$(a + b + c)(a + b + d)(a + c + d)(b + c + d).$$

G14. Consider the system of equations

$$\begin{aligned} |y - 1| &= 4 - |x - 1|, \\ |y| &= \sqrt{|k - x|}. \end{aligned}$$

Find the largest k for which this system has a solution for real values x and y.

G15. You are filling in a 3×3 grid with the numbers $1, 2, \ldots, 9$, such that each number is only used once. Stephanie is happy if any two consecutive numbers are in the same row or the same column. In how many ways can you fill in the 3×3 grid so that Stephanie is happy?

G16. Let $T_n = 1 + 2 + \cdots + n$ denote the nth triangular number. Find the number of positive integers n less than 100 such that n and T_n have the same number of positive integer factors.

G17. Let $ABCD$ be a square, and let P be a point inside it such that $PA = 4$, $PB = 2$, and $PC = 2\sqrt{2}$. What is the area of $ABCD$?

G18. The Fibonacci sequence $\{F_n\}$ is defined as $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all integers $n \geq 0$. Let

$$S = \frac{1}{F_6 + \frac{1}{F_6}} + \frac{1}{F_8 + \frac{1}{F_8}} + \frac{1}{F_{10} + \frac{1}{F_{10}}} + \frac{1}{F_{12} + \frac{1}{F_{12}}} + \cdots.$$

Compute $420S$.

G19. Let $ABCD$ be a square with side length 5. Point P is located inside the square such that the distances from P to AB and AD are 1 and 2 respectively. A point T is selected uniformly at random inside $ABCD$. Let p be the probability that quadrilaterals $APCT$ and $BPDT$ are both not self-intersecting and have areas that add to no more than 10. If p can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n, find $m + n$.

Note: A quadrilateral is self-intersecting if any two of its edges cross.

G20. Sir William has nine knights in his royal army. One day, he decides to send each knight to one of three possible villages in his kingdom. Furthermore, he gives every knight either one or two shields, where all shields are identical, before the knight departs. Let S be the number of ways Sir William can send his knights to the three villages with either one or two shields each such that after all knights arrive at their destination, the number of shields at every village is even. Determine the largest odd factor of S.