MOAA 2020: Gunga Bowl

October 10th, 2020, 4:00PM to 5:00PM Eastern

Gunga Bowl Problems

Gunga Bowl Set 1

B1. Evaluate $2 + 0 - 2 \times 0$.

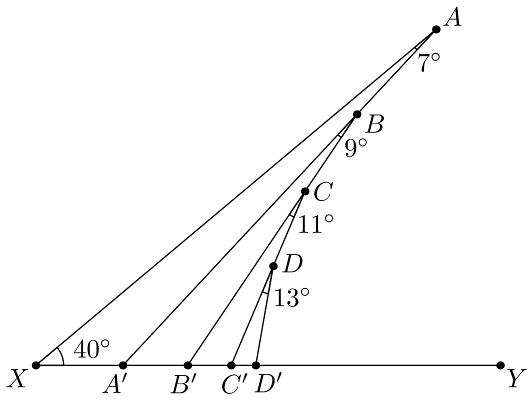
B2. It takes four painters four hours to paint four houses. How many hours does it take forty painters to paint forty houses?

B3. Let a be the answer to this question. What is $\frac{1}{2-a}$?

Gunga Bowl Set 2

B4. Every day at Andover is either *sunny* or *rainy*. If today is sunny, there is a 60% chance that tomorrow is sunny and a 40% chance that tomorrow is rainy. On the other hand, if today is rainy, there is a 60% chance that tomorrow is rainy and a 40% chance that tomorrow is sunny. Given that today is sunny, the probability that the day after tomorrow is sunny can be expressed as $n\%$, where n is a positive integer. What is n?

B5. In the diagram below, what is the value of $\angle DD'Y$ in degrees?



B6. Christina, Jeremy, Will, and Nathan are standing in a line. In how many ways can they be arranged such that Christina is to the left of Will and Jeremy is to the left of Nathan?

Note: Christina does not have to be next to Will and Jeremy does not have to be next to Nathan. For example, arranging them as Christina, Jeremy, Will, Nathan would be valid.

Gunga Bowl Set 3

B7. Let P be a point on side AB of square $ABCD$ with side length 8 such that $PA = 3$. Let Q be a point on side AD such that $PQ \perp PC$. The area of quadrilateral $PQDB$ can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

B8. Jessica and Jeffrey each pick a number uniformly at random from the set $\{1,2,3,4,5\}$ (they could pick the same number). If Jessica's number is x and Jeffrey's number is y, the probability that x^y has a units digit of 1 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

B9. For two points (x_1, y_1) and (x_2, y_2) in the plane, we define the *taxicab distance* between them as

$$|x_1 - x_2| + |y_1 - y_2|.$$

For example, the taxicab distance between $(-1, 2)$ and $(3, \sqrt{2})$ is $6 - \sqrt{2}$. What is the largest number of points Nathan can find in the plane such that the taxicab distance between any two of the points is the same?

Gunga Bowl Set 4

B10. Will wants to insert some $\times$ symbols between the following numbers:

$$1 \quad 2 \quad 3 \quad 4 \quad 6$$

to see what kinds of answers he can get. For example, here is one way he can insert $\times$ symbols:

$$1 \times 23 \times 4 \times 6 = 552.$$

Will discovers that he can obtain the number 276. What is the sum of the numbers that he multiplied together to get 276?

B11. Let $ABCD$ be a parallelogram with $AB = 5$, $BC = 3$, and $\angle BAD = 60^\circ$. Let the angle bisector of $\angle ADC$ meet $\overline{AC}$ at E and $\overline{AB}$ at F. The length EF can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

B12. Find the sum of all positive integers n such that $\lfloor \sqrt{n^2 - 2n + 19} \rfloor = n$.

Note: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Gunga Bowl Set 5

B13. This year, February 29 fell on a Saturday. What is the next year in which February 29 will be a Saturday?

B14. Let $f(x) = \frac{1}{x} - 1$. Evaluate

$$f\left(\frac{1}{2020}\right) \times f\left(\frac{2}{2020}\right) \times f\left(\frac{3}{2020}\right) \times \cdots \times f\left(\frac{2019}{2020}\right).$$

B15. Square $WXYZ$ is inscribed in square $ABCD$ with side length 1 such that W is on AB, X is on BC, Y is on CD, and Z is on DA. Line WY hits AD and BC at points P and R respectively, and line XZ hits AB and CD at points Q and S respectively. If the area of $WXYZ$ is $\frac{13}{18}$, then the area of $PQRS$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. What is $m+n$?

Gunga Bowl Set 6

B16. Let ℓ_r denote the line $x + ry + r^2 = 420$. Jeffrey draws the lines ℓ_a and ℓ_b and calculates their single intersection point. Amazingly, he ends up with the point (a, b)! If a is a positive integer, what is $|b|$?

B17. Let set $\mathcal{L}$ consist of lines of the form $3x + 2ay = 60a + 48$ across all real constants a. For every line ℓ in $\mathcal{L}$, the point on ℓ closest to the origin is in set $\mathcal{T}$. The area enclosed by the locus of all the points in $\mathcal{T}$ can be expressed in the form $n\pi$ for some positive integer n. Compute n.

B18. What is remainder when the 2020-digit number $202020\cdots 20$ is divided by 275?

Gunga Bowl Set 7

B19. Consider right triangle $\triangle ABC$ where $\angle ABC = 90°$, $\angle ACB = 30°$, and $AC = 10$. Suppose a beam of light is shot out from point A. It bounces off side BC and then bounces off side AC, and then hits point B and stops moving. If the beam of light travelled a distance of d, then compute d^2.

B20. Let S be the set of all three digit numbers whose digits sum to 12. What is the sum of all the elements in S?

B21. Consider all ordered pairs (m, n) where m is a positive integer and n is an integer that satisfy

$$m! = 3n^2 + 6n + 15,$$

where $m! = m \times (m - 1) \times \cdots \times 1$. Determine the product of all possible values of n.

Gunga Bowl Set 8

B22. Compute the number of ordered pairs of integers (m, n) satisfying $1000 > m > n > 0$ and

$$6 \cdot \text{lcm}(m - n, m + n) = 5 \cdot \text{lcm}(m, n).$$

B23. Andrew is flipping a coin ten times. After every flip, he records the result (heads or tails). He notices that after every flip, the number of heads he had flipped was always at least the number of tails he had flipped. In how many ways could Andrew have flipped the coin?

B24. Consider a triangle ABC with $AB = 7$, $BC = 8$, and $CA = 9$. Let D lie on $\overline{AB}$ and E lie on $\overline{AC}$ such that $BCED$ is a cyclic quadrilateral and D, O, E are collinear, where O is the circumcenter of ABC. The area of $\triangle ADE$ can be expressed as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. What is $m + n + p$?

Gunga Bowl Set 9

This set consists of three estimation problems, with scoring schemes described.

B25. Submit one of the following ten numbers:

$$3 \quad 6 \quad 9 \quad 12 \quad 15 \quad 18 \quad 21 \quad 24 \quad 27 \quad 30.$$

The number of points you will receive for this question is equal to the number you selected divided by the total number of teams that selected that number, then rounded up to the nearest integer. For example, if you and four other teams select the number 27, you would receive $\lceil \frac{27}{5} \rceil = 6$ points.

B26. Submit any integer from 1 to 1,000,000, inclusive. The *standard deviation* σ of all responses x_i to this question is computed by first taking the arithmetic mean μ of all responses, then taking the square root of average of $(x_i - \mu)^2$ over all i. More, precisely, if there are N responses, then

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2}.$$

For this problem, your goal is to estimate the standard deviation of all responses.

An estimate of e gives $\max\left\{\left\lceil 130 \cdot \left(\min\left\{\frac{\sigma}{e}, \frac{e}{\sigma}\right\}\right)^3\right\rceil - 100, 0\right\}$ points.

B27. For a positive integer n, let $f(n)$ denote the number of distinct nonzero exponents in the prime factorization of n. For example, $f(36) = f(2^2 \times 3^2) = 1$ and $f(72) = f(2^3 \times 3^2) = 2$. Estimate

$$N = f(2) + f(3) + \cdots + f(10000).$$

An estimate of e gives $\max\{30 - \lfloor 7\log_{10}(|N - e|)\rfloor, 0\}$ points.