MOAA 2020: Theme Round

October 10th, 2020, 12:15PM to 1:00PM Eastern

Rules

- No mathematical texts, notes, or online resources of any kind are permitted. Rely on your brain and those of your teammates!
- Compasses, protractors, rulers, straightedges, graph paper, blank scratch paper, and writing implements are generally permitted, so long as they are not designed to give an unfair advantage.
- No computational aids (including but not limited to calculators, phones, calculator watches, and computer programs) are permitted on any portion of the MOAA.
- Individuals may only help or receive help from members of their team — consulting any other individual is grounds for disqualification.

How to Compete

- Participants are recommended to use a video calling software which supports screen sharing such as Zoom to communicate with members of their team. They may also use collaborative whiteboard software such as Miro to share diagrams.
- After completing the test, the team captain should input answers to the problems in the submission portal for their registered team account.

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Theme Round Problems

The Theme Round consists of 20 problems, organized in four **themes** of five problems each, to be solved in 45 minutes. Be sure to try every theme, since problems are ordered by increasing difficulty within themes, but not across themes. For example, the first problems in any theme will all be significantly easier than the last problems in any theme. All answers are nonnegative integers no larger than 1,000,000.

Optimization

Mathematical **optimization** is all about finding the maximum or minimum of a value under a given set of conditions. Try your hand on several optimization problems below!

TO1. What is the maximum number of circles of radius 1 you can fit, without overlapping them, in a circle of radius 3?

TO2. The Den has two deals on chicken wings. The first deal is 4 chicken wings for 3 dollars, and the second deal is 11 chicken wings for 8 dollars. If Jeremy has 18 dollars, what is the largest number of chicken wings he can buy?

TO3. Consider the addition

$$\begin{array}{ccccc} & & \mathrm{O} & \mathrm{N} & \mathrm{E} \\ + & & \mathrm{T} & \mathrm{W} & \mathrm{O} \\ \hline & \mathrm{F} & \mathrm{O} & \mathrm{U} & \mathrm{R} \end{array}$$

where different letters represent different **nonzero** digits. What is the smallest possible value of the four-digit number FOUR?

TO4. Over all real numbers x, let k be the minimum possible value of the expression

$$\sqrt{x^2+9}+\sqrt{x^2-6x+45}.$$

Determine k^2.

TO5. For a real number x, the minimum value of the expression

$$\frac{2x^2+x-3}{x^2-2x+3}$$

can be written in the form $\frac{a-\sqrt{b}}{c}$, where a, b, and c are positive integers such that a and c are relatively prime. Find $a+b+c$.

Relay

Each problem in this section will depend on the previous one! The values $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ refer to the answers to problems 1, 2, 3, and 4, respectively.

TR1. The number 2020 has three different prime factors. What is their sum?

TR2. Let $\mathcal{A}$ be the answer to the previous problem. Suppose ABC is a triangle with $AB = 81$, $BC = \mathcal{A}$, and $\angle ABC = 90^\circ$. Let D be the midpoint of $\overline{BC}$. The perimeter of $\triangle CAD$ can be written as $x + y\sqrt{z}$, where x, y, and z are positive integers and z is not divisible by the square of any prime. What is $x + y$?

TR3. Let $\mathcal{B}$ the answer to the previous problem. What is the unique real value of k such that the parabola $y = \mathcal{B}x^2 + k$ and the line $y = kx + \mathcal{B}$ are tangent?

TR4. Let $\mathcal{C}$ be the answer to the previous problem. How many ordered triples of positive integers (a, b, c) are there such that $\gcd(a, b) = \gcd(b, c) = 1$ and $abc = \mathcal{C}$?

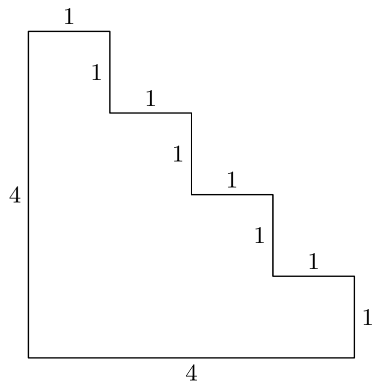
TR5. Let $\mathcal{D}$ be the answer to the previous problem. Let $ABCD$ be a square with side length $\mathcal{D}$ and circumcircle ω. Denote points C' and D' as the reflections over line AB of C and D respectively. Let P and Q be the points on ω, with A and P on opposite sides of line BC and B and Q on opposite sides of line AD, such that lines $C'P$ and $D'Q$ are both tangent to ω. If the lines AP and BQ intersect at T, what is the area of $\triangle CDT$?

to Bash or not to Bash...

In math competitions, we call a problem "bashy" if it involves a lot of routine yet annoying computations and/or casework. The following problems all seem bashy at first glance, but we promise they all have nicer solutions that don't require too much computation.

Do you have the ingenuity to find the fast and clever approaches? Or would you prefer to spend the extra time and effort to simply bash out the answer? To bash or not to bash, that is your question to answer.

TB1. Find the perimeter of the figure below, where all angles are right angles.



TB2. Evaluate

$$1 \times 5 + 2 \times 8 + 3 \times 13 + 5 \times 21 + 8 \times 34 + 13 \times 55.$$

TB3. Jeff and Geoff each choose a random number from the set $\{1, 2, \ldots, 100\}$. Let Jeff's number be a and Geoff's number be b. Given that $a \neq b$, the probability that $ab + a + b$ is odd can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

TB4. Find the number of positive integers n less than 100 that satisfy the equation

$$n = \lfloor \sqrt{n} \rfloor \cdot \lceil \sqrt{n} \rceil.$$

Note: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the least integer greater than or equal to x.

TB5. Arnav writes every positive integer factor of 2020^2 exactly once on a blackboard. Every minute, he chooses a number on the blackboard uniformly at random, and he erases it as well as all of its factors. The expected amount of minutes that Arnav takes to erase every number on the board can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Functions

The following problems all involve functions. A function can be thought of as a process that takes in some input and spits out some output. For example, $f(x) = x^2 - 1$ is a simple function which takes in 3 and spits out $f(3) = 3^2 - 1 = 8$.

TF1. We define the *prime-counting function* $\pi(n)$ as the number of primes that are less than or equal to n. For example, $\pi(8) = 4$ since there are 4 primes less than or equal to 8, namely, 2, 3, 5, and 7. What is the sum of all positive integers n with $\pi(n) = 6$?

Note: The π used here is unrelated to the infinite decimal $3.1415\ldots$

TF2. Consider the function $f(x) = 2020 - x$. Find $f(f(f(f(f(1)))))$.

TF3. Consider the polynomial $P(x)$ such that for any positive real number x, $P(x)$ equals the numerical sum of the volume and surface area of regular tetrahedron with side length x. If r is the only nonzero real root of this polynomial, determine r^2.

TF4. Let $P(x) = a_5x^5 + a_4x^4 + \cdots + a_1x + a_0$ be the unique polynomial of degree 5 satisfying

$$\begin{aligned} P(0) &= 1, \\ P(1) &= 2, \\ P(2) &= 4, \\ P(3) &= 8, \\ P(4) &= 16, \\ P(5) &= 32. \end{aligned}$$

Given that there is a unique positive integer n such that $P(n) = 1024$, find n.

TF5. Determine the number of bijections f on the set $\{1, 2, \ldots, 8\}$ satisfying

$$|2f(i) - 2i - 1| \leq 3,$$

for all $i = 1, 2, \ldots, 8$.

Note: A bijection on a finite set X is a function from X to itself such that for any $a, b \in X$ with $a \neq b$, then $f(a) \neq f(b)$.