MOAA 2020: General Round Solutions

October 10, 2020

G1. What is $20 \times 20 - 19 \times 19$?

Proposed by: William Yue

Answer: $\boxed{39}$

Solution: By difference of squares, $20^2 - 19^2 = (20 - 19)(20 + 19) = 39$.

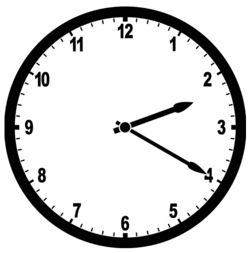
G2. Andover has a total of 1440 students and teachers as well as a $1 : 5$ teacher-to-student ratio (for every teacher, there are exactly 5 students). In addition, every student is either a boarding student or a day student, and 70% of the students are boarding students. How many day students does Andover have?

Proposed by: William Yue

Answer: $\boxed{360}$

Solution: Since there is a $1 : 5$ teacher-to-student ratio, there must be $1440 \times \frac{5}{6} = 1200$ students. We know 70% of the students are boarding students, so 30% of the students must be day students. Our answer is then $1200 \times \frac{3}{10} = 360$.

G3. The time is 2:20. If the acute angle between the hour hand and the minute hand of the clock measures x degrees, find x.



Proposed by: William Yue

Answer: $\boxed{50}$

Solution: Label the hour hand as ℓ_1 and the minute hand as ℓ_2. We seek the acute angle between ℓ_1 and ℓ_2. Let ℓ be the line passing through the clock's center and the number 2 on the clock. Then, the acute angle between ℓ and ℓ_2 is just $360° \times \frac{2}{12} = 60°$. Since we are at 20 minutes past the hour, the hour hand must be $\frac{20}{60} = \frac{1}{3}$ of the way between 2 and 3. Thus, the acute angle between ℓ and ℓ_1 measures $360° \times \frac{1}{12} \times \frac{1}{3} = 10°$. Subtracting these two, we find that the answer is $60° - 10° = 50°$.

G4. Point P is located on segment AC of square $ABCD$ with side length 10 such that $AP > CP$. If the area of quadrilateral $ABPD$ is 70, what is the area of $\triangle PBD$?

Proposed by: Andrew Wen

Answer: $\boxed{20}$

Solution: The area of $\triangle ABD$ is $\frac{1}{2} \times 10 \times 10 = 50$. Then, the area of $\triangle PBD$ is just $70 - 50 = 20$.

G5. Andrew always sweetens his tea with sugar, and he likes a $1 : 7$ sugar-to-unsweetened tea ratio. One day, he makes a 100ml cup of unsweetened tea but realizes that he has run out of sugar. Andrew decides to borrow his sister's jug of pre-made SUPERSWEET tea, which has a $1 : 2$ sugar-to-unsweetened tea ratio. How much SUPERSWEET tea, in ml, does Andrew need to add to his unsweetened tea so that the resulting tea is his desired sweetness?

Proposed by: Jessica He

Answer: $\boxed{60}$

Solution: Suppose we add x ml of SUPERSWEET tea. Then, we are adding $\frac{x}{3}$ ml of sugar and $\frac{2x}{3}$ ml of unsweetened tea. Since we started with 100ml of unsweetened tea, in total, we have $100 + \frac{2x}{3}$ ml of unsweetened tea. Andrew's preferred sugar-to-unsweetened tea ratio asks that

$$\begin{aligned} 100 + \frac{2x}{3} &= 7 \times \frac{x}{3} \\ 100 &= \frac{5x}{3} \\ x &= 60. \end{aligned}$$

G6. Jeremy the architect has built a railroad track across the equator of his spherical home planet which has a radius of exactly 2020 meters. He wants to raise the entire track 6 meters off the ground, everywhere around the planet. In order to do this, he must buy more track, which comes from his supplier in bundles of 2 meters. What is the minimum number of bundles he must purchase? Assume the railroad track was originally built on the ground.

Proposed by: William Yue

Answer: $\boxed{19}$

Solution: Initially, our railroad track forms a circle with radius 2020. The length of this track is 4040π meters. By raising the track above the ground, the radius of the circle increases to 2026. Now, the length of the track is 4052π meters. So, the extra track needed measures 12π meters. Since the track can only be bought in bundles of 2 meters, we need to buy at least $\lceil \frac{12\pi}{2} \rceil = 19$ bundles.

G7. Mr. DoBa writes the numbers $1, 2, 3, \ldots, 20$ on the board. Will then walks up to the board, chooses two of the numbers, and erases them from the board. Mr. DoBa remarks that the average of the remaining 18 numbers is exactly 11. What is the maximum possible value of the larger of the two numbers that Will erased?

Proposed by: Nathan Xiong

Answer: $\boxed{11}$

Solution: Before Will erases two numbers, the sum of all numbers on the board is $1+2+\cdots+20 = \frac{20\cdot 21}{2} = 210$. After Will erases two numbers, the sum of all numbers on the board is $18 \times 11 = 198$. Thus, the sum of the two numbers Will erased is $210 - 198 = 12$.

In order to maximize the larger number Will chose, we want to minimize the smaller number he chose. If the smaller number is 1, the larger number is 11, which is our answer.

G8. Nathan is thinking of a number. His number happens to be the smallest positive integer such that if Nathan doubles his number, the result is a perfect square, and if Nathan triples his number, the result is a perfect cube. What is Nathan's number?

Proposed by: Nathan Xiong

Answer: $\boxed{72}$

Solution: Let Nathan's number be n. We know that $2n = a^2$ and $3n = b^3$ for two positive integers a and b. From the second equation, we have $3 \mid b^3 \implies 3 \mid b$. Clearly, to minimize n, we want to minimize b. The smallest value that b can take is 3. In that case, $n = 9$, and the first equation has no solution in a. The next smallest value that b can take is 6. Then, $n = 72$, and $2n = 144$ is indeed a square number. Hence, our answer is 72.

G9. Let S be the set of positive integers whose digits are in strictly increasing order when read from left to right. For example, 1, 24, and 369 are all elements of S, while 20 and 667 are not. If the elements of S are written in increasing order, what is the 100th number written?

Proposed by: Nathan Xiong

Answer: $\boxed{356}$

Solution: Call a number *upbeat* if its digits are strictly increasing. We want to find the 100th upbeat number.

Clearly, all 1 digit numbers are upbeat. So, there are 9 upbeat numbers here.

For 2 digit numbers, there are $\binom{9}{2} = 36$ upbeat numbers. To prove this, imagine choosing any two distinct nonzero digits (upbeat numbers cannot have a 0 in them) and placing them in increasing order. This gives an upbeat number, and any upbeat number can be found this way.

Now, we consider 3 digit numbers starting with 1. Using the same argument as the one for 2 digit upbeat numbers, we find that there are $\binom{8}{2} = 28$ upbeat numbers here.

Similarly, for 3 digit numbers starting with 2, there are $\binom{7}{2} = 21$ upbeat numbers.

So far, we have listed $9 + 36 + 28 + 21 = 94$ upbeat numbers. At this point, we can just list the next few upbeat numbers to find the answer. The 100th upbeat number happens to be 356.

G10. Find the largest prime factor of the expression

$$2^{20} + 2^{16} + 2^{12} + 2^8 + 2^4 + 1.$$

Proposed by: Nathan Xiong

Answer: $\boxed{241}$

Solution: For convenience, let $x = 2^4$. Our expression is then $x^5 + x^4 + x^3 + x^2 + x + 1$. The key idea is the factorization

$$\begin{aligned} x^5 + x^4 + x^3 + x^2 + x + 1 &= \frac{x^6 - 1}{x - 1} \\ &= \frac{(x^2 - 1)(x^4 + x^2 + 1)}{(x - 1)} \\ &= \frac{(x - 1)(x + 1)(x^4 + x^2 + 1)}{x - 1} \\ &= (x + 1)(x^4 + x^2 + 1) \\ &= (x + 1)(x^2 + x + 1)(x^2 - x + 1). \end{aligned}$$

Now, we can check $x + 1 = 17$, $x^2 + x + 1 = 273$, and $x^2 - x + 1 = 241$. We can also check that $273 = 3 \times 7 \times 13$ and that 241 is prime. Hence, our answer is 241.

G11. Christina writes down all the numbers from 1 to 2020, inclusive, on a whiteboard. What is the sum of all the digits that she wrote down?

Proposed by: William Yue

Answer: $\boxed{28144}$

Solution: We consider the contribution from each digit separately.

First, consider the thousands digit. It is 0 for 999 numbers, 1 for 1000 numbers, and 2 for 21 numbers. Thus, the total contribution from the thousands digit is $1 \times 1000 + 2 \times 21 = 1042$.

Next, consider the hundreds digit. For the numbers at least 2000, this is 0. For numbers less than 2000, each digit shows up an equal number of times. Thus, each digit shows up $\frac{2000}{10} = 200$ times. Hence, the total contribution from the hundreds digit is $(1 + 2 + \cdots + 9) \times 200 = 9000$.

The case for the tens digit is similar to that of the hundreds digit, except that we also have to count an extra 12 for the numbers bigger than 2000. The total contribution from the tens digit is $9000 + 12 = 9012$.

Finally, the case for the ones digit is similar to that of the hundreds digit, except that we also have to count an extra $(1 + 2 + \cdots + 9) \times 2 = 90$ for the numbers bigger than 2000. The total contribution from the ones digit is $9000 + 90 = 9090$.

The answer is $1042 + 9000 + 9012 + 9090 = 28144$.

G12. Triangle ABC has side lengths $AB = AC = 10$ and $BC = 16$. Let M and N be the midpoints of segments BC and CA, respectively. There exists a point $P \neq A$ on segment AM such that $2PN = PC$. What is the area of $\triangle PBC$?

Proposed by: Andrew Wen

Answer: $\boxed{16}$

Solution: Let Q denote the reflection of P over N. Since the diagonals of $APCQ$ bisect each other, $APCQ$ is a parallelogram. Also, our length condition becomes $PQ = PC$, so if we let T denote the midpoint of segment QC, we have $PM = TC = \frac{QC}{2} = \frac{AP}{2}$.

By the Pythagorean Theorem, $AM = 6$, so $PM = 2$, and the area of $\triangle PBC$ is $\frac{1}{2} \times 16 \times 2 = 16$.

G13. Consider the polynomial

$$P(x) = x^4 + 3x^3 + 5x^2 + 7x + 9.$$

Let its four roots be a, b, c, d. Evaluate the expression

$$(a + b + c)(a + b + d)(a + c + d)(b + c + d).$$

Proposed by: Nathan Xiong

Answer: $\boxed{33}$

Solution: Since a, b, c, d are roots, we can write $P(x) = (x - a)(x - b)(x - c)(x - d)$. By Vieta's formulas, $a + b + c + d = -3$. Substituting $x = a + b + c + d$ gives

$$\begin{aligned} 33 &= P(-3) \\ &= P(a + b + c + d) \\ &= (a + b + c)(a + b + d)(a + c + d)(b + c + d). \end{aligned}$$

G14. Consider the system of equations

$$|y - 1| = 4 - |x - 1|,$$
$$|y| = \sqrt{|k - x|}.$$

Find the largest k for which this system has a solution for real values x and y.

Proposed by: Jeremy Zhou

Answer: $\boxed{26}$

Solution: We solve this problem with graphing.

First, we claim that $|y - 1| = 4 - |x - 1|$ is a diamond (a square rotated 45° about its center) with center $(1, 1)$ and side length $4\sqrt{2}$. This can be checked with casework.

- Case 1: $x - 1 \geq 0$ and $y - 1 \geq 0$.

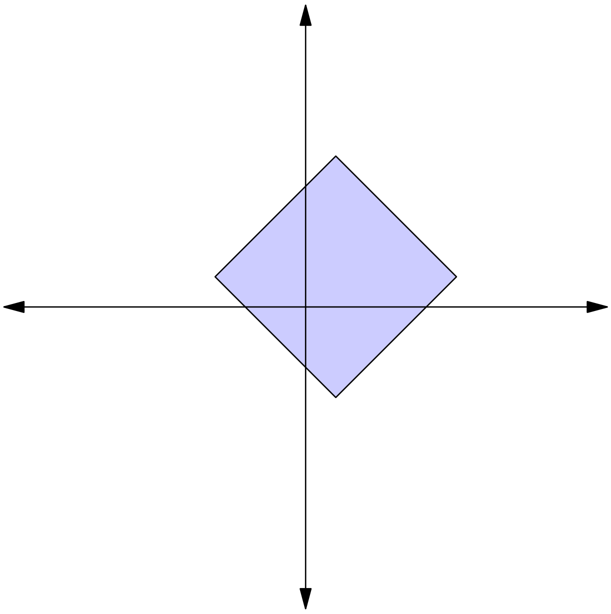
  This is the segment of the line $y = -x + 6$ between $(1, 5)$ and $(5, 1)$.
- Case 2: $x - 1 \geq 0$ and $y - 1 < 0$.

  This is the segment of the line $y = x - 4$ between $(1, -3)$ and $(5, 1)$.
- Case 3: $x - 1 < 0$ and $y - 1 \geq 0$.

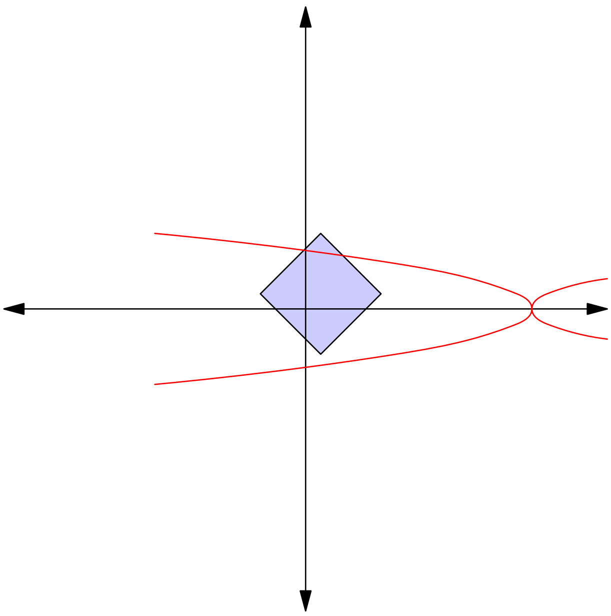
  This is the segment of the line $y = x + 4$ between $(-3, 1)$ and $(1, 5)$.
- Case 4: $x - 1 < 0$ and $y - 1 < 0$.

  This is the segment of the line $y = -x - 2$ between $(-3, 1)$ and $(1, -3)$.

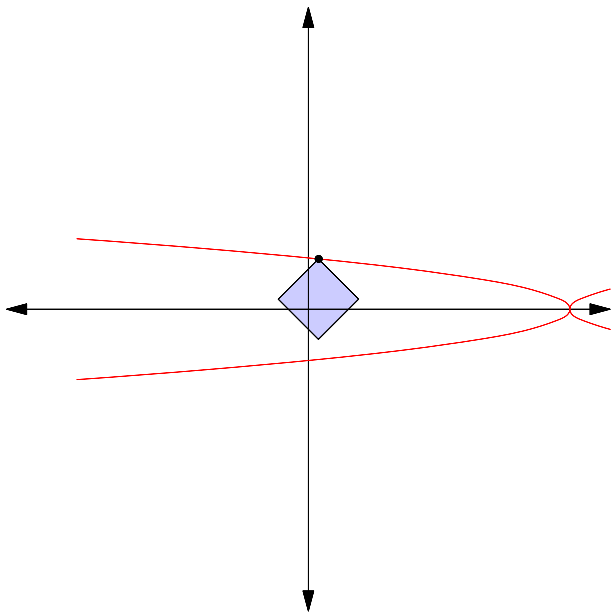
Putting these four cases together gives the diamond shown below:



Now, we consider the graph for the second equation. Squaring both sides gives $y^2 = |k-x|$. The graph of $y^2 = x$ is a horizontal parabola stretching out in the positive x direction. When we change this to $y^2 = |x|$, the graph becomes a double parabola, stretching out in the positive and negative x directions. Finally, the graph of $y^2 = |k-x|$ is just this double parabola shifted to the right by k units. For example, here is the graph when $k = 15$:



Finally, notice that when k is too large, the diamond will lie entirely within the "left arm" of the double parabola. Thus, we want to find the k when the upper corner of the diamond is just touching the double parabola, as shown in this diagram:



In that case, the point $(1, 5)$ lies on the double parabola, so $5^2 = |k - 1|$ and $k = 26$.

G15. You are filling in a 3×3 grid with the numbers $1, 2, \ldots, 9$, such that each number is only used once. Stephanie is happy if any two consecutive numbers are in the same row or the same column. In how many ways can you fill in the 3×3 grid so that Stephanie is happy?

Proposed by: Nathan Xiong

Answer: $\boxed{1512}$

Solution: First, observe that swapping any two rows or columns of the 3×3 grid will not change Stephanie from happy to unhappy or vice versa. Thus, swap rows and columns until the number 1 is in the top left corner. Now, for Stephanie to be happy, the number 2 must be in either the top row or the leftmost column. Swap rows and columns until the number 2 is adjacent and to the right of the number 1. It suffices to find the number of configurations that make Stephanie happy with this initial setup and then multiply by 36, as there are 9 possible starting places for the number 1 and 4 possible starting places for the number 2. The remainder of the solution is casework.

- Case 1: The number 3 is in the top row.

  The number 4 must be in one of two spots in the rightmost column. It doesn't matter which spot it's in as we can just swap the rows, so assume without loss of generality that it's adjacent to the number 3.

  - Subcase 1: The number 5 is in the rightmost column.

    It's easy to count that there are 4 valid configurations in this case.

  - Subcase 2: The number 5 is in the middle row and the leftmost column.

    If the number 6 is also in the middle row, we get 2 valid configurations. If the number 6 is in the leftmost column, we get 1 valid configuration.

  - Subcase 3: The number 5 is in the middle row and the middle column.

    If the number 6 is also in the middle row, we get 2 valid configurations. If the number 6 is in the middle column, we get 1 valid configuration.

  In total, there are $2 \times (4 + 2 + 1 + 2 + 1) = 20$ valid configurations in Case 1.

- Case 2: The number 3 is in the middle column.

  The number 3 must be in one of the two spots in the middle column. It doesn't matter which spot it's in as we can just swap the rows, so assume without loss of generality that it's adjacent to the number 2.

  - Subcase 1: The number 4 is in the leftmost column.

    If the number 5 is also in the leftmost column, we get 2 valid configurations. If the number 5 is in the middle row, we get 2 valid configurations.

  - Subcase 2: The number 4 is in the middle column.

    If the number 5 is in the leftmost column, we get 3 valid configurations. If the number 5 is in the rightmost column, we get 2 valid configurations.

  - Subcase 3: The number 4 is in the rightmost column.

    It's easy to count that there are 2 valid configurations in this case.

  In total, there are $2 \times (2+2+3+2+2) = 22$ valid configurations in Case 2.

The answer is $36 \times (20+22) = 1512$.

G16. Let $T_n = 1 + 2 + \cdots + n$ denote the nth triangular number. Find the number of positive integers n less than 100 such that n and T_n have the same number of positive integer factors.

Proposed by: Nathan Xiong

Answer: $\boxed{14}$

Solution: First, note that $n = 1$ works. Henceforth, assume that $n > 1$.

I claim that n must be even. Suppose that $n > 1$ is odd and that n and $T_n = \frac{n(n+1)}{2}$ have the same number of factors. Note that $\frac{n+1}{2}$ is some integer greater than 1, so n properly divides T_n, which means they can't have the same number of factors.

Next, I claim that $n+1$ must be prime. Suppose that $n+1$ is composite and that n and $T_n = \frac{n(n+1)}{2}$ have the same number of factors. Let $2^{e_1}p_2^{e_2}\cdots p_i^{e_i}$ be the prime factorization of n, and let $q_1^{f_1}q_2^{f_2}\cdots q_j^{f_j}$ be the prime factorization of $n+1$, where $j > 1$. Since n and $n+1$ are relatively prime, none of the primes in the set $\{q_1, q_2, \ldots, q_j\}$ are in the set $\{2, p_2, \ldots, p_i\}$. Thus, we can write

$$T_n = 2^{e_1 - 1}p_2^{e_2}\cdots p_i^{e_i}q_1^{f_1}\cdots q_j^{f_j}.$$

If T_n is to have the same number of factors as n, we must have

$$(e_1+1)(e_2+1)\cdots(e_i+1) = e_1(e_2+1)\cdots(e_i+1)(f_1+1)\cdots(f_j+1).$$

Cancelling out terms, we get

$$e_1 + 1 = e_1(f_1+1)\cdots(f_j+1).$$

However, since $j > 1$, $(f_1+1)\cdots(f_j+1) \geq (1+1)(1+1) = 4$. The above equation then implies $e_1 + 1 \geq 4e_1$, which is impossible. Hence, $n+1$ must be prime.

Thus, suppose that $n+1 = r$ for a prime r. Using our equation above, we need $e_1 + 1 = 2e_1$, so $e_1 = 1$. Hence, we can write $n = 2(2m-1)$ for a positive integer m. Then, $n+1 = r = 4m-1$, so $r \equiv 3 \pmod 4$.

Finally, it's easy to see that all numbers n that are 1 less than a prime that is 3 (mod 4) work. In total, there are 14 numbers that work: 1, 2, 6, 10, 18, 22, 30, 42, 46, 58, 66, 70, 78, and 82.

G17. Let $ABCD$ be a square, and let P be a point inside it such that $PA = 4$, $PB = 2$, and $PC = 2\sqrt{2}$. What is the area of $ABCD$?

Proposed by: Nathan Xiong

Answer: $\boxed{20}$

Solution: Rotate $\triangle BCP$ by 90° counterclockwise around B, so that C aligns with A. Suppose P gets sent to point Q. Since $\angle PBQ = 90^\circ$ and $BP = BQ$, $\triangle PBQ$ is a right isosceles triangle. Hence, $QP = 2\sqrt{2}$. Furthermore, we know $QA = PC = 2\sqrt{2}$, $QP = 2\sqrt{2}$, and $PA = 4$. By the Pythagorean Theorem, $\angle AQP = 90^\circ$, and $\triangle AQP$ is a right isosceles triangle.

Now, $\angle APB = \angle APQ + \angle QPB = 45^\circ + 45^\circ = 90^\circ$. By the Pythagorean Theorem, the side length of square $ABCD$ is $\sqrt{4^2 + 2^2} = \sqrt{20}$, and the area is 20.

G18. The Fibonacci sequence $\{F_n\}$ is defined as $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all integers $n \geq 0$. Let

$$S = \frac{1}{F_6 + \frac{1}{F_6}} + \frac{1}{F_8 + \frac{1}{F_8}} + \frac{1}{F_{10} + \frac{1}{F_{10}}} + \frac{1}{F_{12} + \frac{1}{F_{12}}} + \cdots.$$

Proposed by: Nathan Xiong

Answer: $\boxed{84}$

Solution: First, simplify each fraction in the sum.

$$S = \frac{F_6}{F_6^2 + 1} + \frac{F_8}{F_8^2 + 1} + \frac{F_{10}}{F_{10}^2 + 1} + \frac{F_{12}}{F_{12}^2 + 1} + \cdots.$$

The key claim is the following identity, commonly known as Cassini's Identity.

Lemma. For every even number n, we have $F_n^2 + 1 = F_{n-1}F_{n+1}$.

Proof. We use induction on n. For the base case, it's easy to check that $F_2^2 + 1 = F_1F_3$.

Now, assume that the identity holds for some even number k. Then,

$$\begin{aligned}
F_{k+2}^2 + 1 &= (F_k + F_{k+1})^2 + 1 \\
&= F_k^2 + 1 + 2F_kF_{k+1} + F_{k+1}^2 \\
&= F_{k-1}F_{k+1} + 2F_kF_{k+1} + F_{k+1}^2 \\
&= F_{k+1}(F_{k-1} + F_k + F_k + F_{k+1}) \\
&= F_{k+1}(F_{k+1} + F_{k+2}) \\
&= F_{k+1}F_{k+3}.
\end{aligned}$$

This completes the induction. □

Using this lemma, we can write

$$\begin{aligned}
S &= \frac{F_6}{F_5F_7} + \frac{F_8}{F_7F_9} + \frac{F_{10}}{F_9F_{11}} + \frac{F_{12}}{F_{11}F_{13}} + \cdots \\
&= \frac{F_7 - F_5}{F_5F_7} + \frac{F_9 - F_7}{F_7F_9} + \frac{F_{11} - F_9}{F_9F_{11}} + \frac{F_{13} - F_{11}}{F_{11}F_{13}} + \cdots \\
&= \left(\frac{1}{F_5} - \frac{1}{F_7}\right) + \left(\frac{1}{F_7} - \frac{1}{F_9}\right) + \left(\frac{1}{F_9} - \frac{1}{F_{11}}\right) + \left(\frac{1}{F_{11}} - \frac{1}{F_{13}}\right) + \cdots \\
&= \frac{1}{F_5} \\
&= \frac{1}{5}.
\end{aligned}$$

The answer is $\frac{420}{5} = 84$.

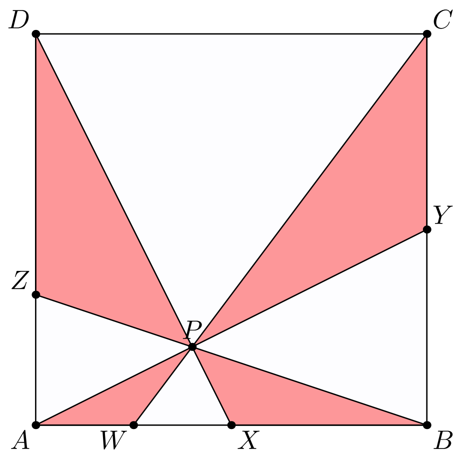
G19. Let $ABCD$ be a square with side length 5. Point P is located inside the square such that the distances from P to AB and AD are 1 and 2 respectively. A point T is selected uniformly at random inside $ABCD$. Let p be the probability that quadrilaterals $APCT$ and $BPDT$ are both not self-intersecting and have areas that add to no more than 10. If p can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n, find $m+n$.

Note: A quadrilateral is self-intersecting if any two of its edges cross.

Proposed by: Andrew Wen

Answer: $\boxed{151}$

Solution:



Consider the diagram above. We impose Cartesian coordinates. Let $A = (0,0)$, $B = (5,0)$, $C = (5,5)$, $D = (0,5)$, and $P = (2,1)$.

Consider the following:

- If T is in the bottom left red region, then CT will intersect AP, so $APCT$ is self-intersecting.
- If T is in the bottom right red region, then DT will intersect BP, so $BPDT$ is self-intersecting.
- If T is in the top right red region, then AT will intersect CP, so $APCT$ is self-intersecting.
- If T is in the top left red region, then BT will intersect DP, so $BPDT$ is self-intersecting.

Hence, T cannot be in any red region. It is easy to check that if T is not in any red region, then $APCT$ and $BPDT$ are not self-intersecting. Hence, ignoring the area condition, T can be located in any non-red region.

Next, we decipher the area condition. Let $T = (x, y)$. By the Shoelace formula,

$$[APCT] = \frac{1}{2}|10 + 5y - 5 - 5x| = \frac{5}{2}|1 - x + y|,$$

$$[BPDT] = \frac{1}{2}|5 + 10 - 5x - 5y| = \frac{5}{2}|3 - x - y|,$$

which means that we need $|1 - x + y| + |3 - x - y| \leq 4$. We do some casework to find what graph this details.

- Case 1: $x - y < 1$ and $x + y < 3$.

  This is the part of $x \geq 0$ belonging to both $y > x - 1$ and $y < -x + 3$. Hence, it determines a triangle with vertices $(0, 3)$, $(2, 1)$, and $(0, -1)$.

- Case 2: $x - y < 1$ and $x + y \geq 3$.

  This is the part of $y \leq 3$ belonging to both $y > x - 1$ and $y \geq -x + 3$. Hence, it determines a triangle with vertices $(0, 3)$, $(4, 3)$, and $(2, 1)$.

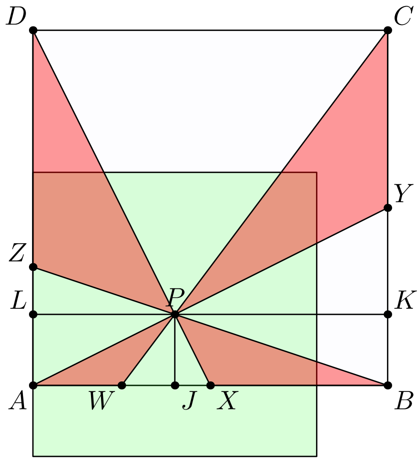
- Case 3: $x - y \geq 1$ and $x + y < 3$.

  This is the part of $y \geq -1$ belonging to both $y \leq x - 1$ and $y < -x + 3$. Hence, it determines a triangle with vertices $(4, -1)$, $(0, -1)$, and $(2, 1)$.

- Case 4: $x - y \geq 1$ and $x + y \geq 3$.

  This is the part of $x \leq 4$ belonging to both $y \leq x - 1$ and $y \geq -x + 3$. Hence, it determines a triangle with vertices $(4, -1)$, $(2, 1)$, and $(4, 3)$.

Putting these four triangles together, we get a square with center $(2, 1)$ and side length 4, as shown by the green shape in the diagram below.



So, it remains to find the area of the set of points inside $ABCD$ that are inside the green square and outside of the red regions. This area consists of four green triangles.

First, consider the left and right green triangles. The height for both triangles is obviously 2. To find the base of the left triangle, just note by similar triangles that $ZA = \frac{ZA}{PJ} = \frac{AB}{BJ} = \frac{5}{3}$. As for the right triangle, we can also use similar triangles to find that it is equal to $\frac{2}{3} \times BY = \frac{2}{3} \times \frac{BY}{PJ} = \frac{2}{3} \times \frac{AB}{AJ} = \frac{2}{3} \times \frac{5}{2} = \frac{5}{3}$. Thus, the area of both the left and the right green triangle is $\frac{1}{2} \times 2 \times \frac{5}{3} = \frac{5}{3}$.

Next, consider the bottom green triangle. Its height is obviously 1. As for the base, note that by similar triangles, we have $\frac{AX}{2} = \frac{AX}{PL} = \frac{AD}{DL} = \frac{5}{4}$ and $\frac{BW}{3} = \frac{BW}{PK} = \frac{BC}{CK} = \frac{5}{4}$. Thus, $AX = \frac{5}{2}$ and $BW = \frac{15}{4}$, which implies $WX = \frac{5}{2} + \frac{15}{4} - 5 = \frac{5}{4}$. Thus, the area of the bottom green triangle is $\frac{1}{2} \times 1 \times \frac{5}{4} = \frac{5}{8}$.

Finally, the top green triangle is similar to the bottom triangle with similarity ratio 2. Thus, the area of the top green triangle is $4 \times \frac{5}{8} = \frac{5}{2}$.

Hence, the total area of the region of points T could be in is

$$\frac{5}{3} + \frac{5}{3} + \frac{5}{8} + \frac{5}{2} = \frac{155}{24}.$$

Our probability is therefore $p = \frac{155}{24 \times 25} = \frac{31}{120}$, and the answer is $31 + 120 = 151$.

G20. Sir William has nine knights in his royal army. One day, he decides to send each knight to one of three possible villages in his kingdom. Furthermore, he gives every knight either one or two shields, where all shields are identical, before the knight departs. Let S be the number of ways Sir William can send his knights to the three villages with either one or two shields each such that after all knights arrive at their destination, the number of shields at every village is even. Determine the largest odd factor of S.

Proposed by: Andrew Wen

Answer: $\boxed{10611}$

Solution: First, note that the knights that are given two shields don't affect the parity of the number of shields at any village. Also, if the number of shields at each village is even, then the number of knights who received one shield at each village must be even. In particular, the total number of knights given one shield must be even.

Suppose that n knights receive one shield, where n is even. There are $\binom{9}{n}$ ways to choose these n knights and 3^{9-n} ways of assigning the $9-n$ other knights that are given two shields to the three possible locations. It remains to split the n knights who receive one shield among the three villages such that each village gets an even number of knights.

Lemma. The number of ways to split n knights, where n is even, into three groups such that each group has an even number of knights is $\frac{3^n+3}{4}$.

Proof. Consider the expansion of $(x+y+z)^n$. Each term represents one of the 3^n total possible splittings where the exponent of x represents the number of knights in the first group, the exponent of y represents the number of knights in the second group, and the exponent of z represents the number of knights in the third group.

We wish to compute the sum of coefficients for all terms where all the exponents are even. This can be done with a clever "filter" argument. The desired sum can be written as

$$\frac{1}{8}\sum_{(x,y,z)\in\{\pm1\}^3}(x+y+z)^n = \frac{3^n+3}{4},$$

as desired. □

Thus, the sum S is

$$\sum_{\substack{0\le n\le 9;\\ n \text{ even}}}\binom{9}{n}\cdot 3^{9-n}\cdot\frac{3^n+3}{4} = \frac{1}{4}\sum_{\substack{0\le n\le 9;\\ n \text{ even}}}\binom{9}{n}\cdot(3^9+3^{10-n}).$$

Upon simplification using $\sum_{\substack{0\le n\le 9;\\ n \text{ even}}}\binom{9}{n} = 2^8$ and a similar filter argument to the one used above, the sum S becomes

$$\begin{aligned}\frac{1}{4}\cdot 2^8\cdot 3^9+\frac{3}{4}\sum_{\substack{0\le n\le 9;\\ n \text{ odd}}}\binom{9}{n}\cdot 3^n &= 2^6\times 3^9+\frac{3}{4}\left(\frac{4^9+2^9}{2}\right)\\ &= 2^6\times 3^9+3\times(2^{15}+2^6)\\ &= 2^6\times(3^9+3\times 2^9+3)\\ &= 2^7\times 10611.\end{aligned}$$

Hence, the answer is 10611.

Remark: The filtering trick used in the solution above is a special case of the *roots of unity filter*.