MOAA 2020: Gunga Bowl Solutions

October 10, 2020

Gunga Bowl Set 1

B1. Evaluate $2 + 0 - 2 \times 0$.

Proposed by: Nathan Xiong

Answer: $\boxed{2}$

Solution: Just use order of operations: $2 + 0 - 2 \times 0 = 2 - 0 = 2$.

B2. It takes four painters four hours to paint four houses. How many hours does it take forty painters to paint forty houses?

Proposed by: Nathan Xiong

Answer: $\boxed{4}$

Solution: Split the painters such that each painter paints one house, and they all work simultaneously. Then, one painter paints one house in four hours. Hence, forty painters paint forty houses in four hours.

B3. Let a be the answer to this question. What is $\frac{1}{2-a}$?

Proposed by: Nathan Xiong

Answer: $\boxed{1}$

Solution: The problem tells us that $a = \frac{1}{2-a}$. Multiplying both sides by $2 - a$, we have $2a - a^2 = 1$, which rearranges to $(a - 1)^2 = 0$. Hence, $a = 1$.

Gunga Bowl Set 2

B4. Every day at Andover is either *sunny* or *rainy*. If today is sunny, there is a 60% chance that tomorrow is sunny and a 40% chance that tomorrow is rainy. On the other hand, if today is rainy, there is a 60% chance that tomorrow is rainy and a 40% chance that tomorrow is sunny. Given that today is sunny, the probability that the day after tomorrow is sunny can be expressed as $n\%$, where n is a positive integer. What is n?

Proposed by: Nathan Xiong

Answer: $\boxed{52}$

Solution: There are two cases to consider.

- Case 1: Tomorrow is sunny.

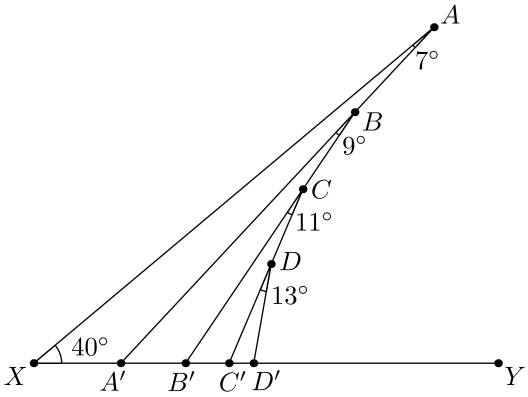
  The probability that tomorrow is sunny is 60%. Then, given that tomorrow is sunny, the probability that the day after tomorrow is sunny is 60%. Hence, the probability of this case is $60\% \times 60\% = 36\%$.

- Case 2: Tomorrow is rainy.

  The probability that tomorrow is rainy is 40%. Then, given that tomorrow is rainy, the probability that the day after tomorrow is sunny is 40%. Hence, the probability of this case is $40\% \times 40\% = 16\%$.

Thus, the total probability is $36\% + 16\% = 52\%$, and the answer is 52.

B5. In the diagram below, what is the value of $\angle DD'Y$ in degrees?



Proposed by: Nathan Xiong

Answer: $\boxed{80}$

Solution: Repeatedly apply the Exterior Angle theorem:

$$\angle BA'B' = \angle AXA' + \angle XAA' = 47^\circ$$
$$\angle CB'C' = \angle BA'B' + \angle A'BB' = 56^\circ$$
$$\angle DC'D' = \angle CB'C' + \angle B'CC' = 67^\circ$$
$$\angle DD'Y = \angle DC'D' + \angle C'DD' = 80^\circ$$

B6. Christina, Jeremy, Will, and Nathan are standing in a line. In how many ways can they be arranged such that Christina is to the left of Will and Jeremy is to the left of Nathan?

Note: Christina does not have to be next to Will and Jeremy does not have to be next to Nathan. For example, arranging them as Christina, Jeremy, Will, Nathan would be valid.

Proposed by: Nathan Xiong

Answer: $\boxed{6}$

Solution: In an arbitrary permutation of Christina, Jeremy, Will, and Nathan, there is a $\frac{1}{2}$ probability that Christina is to the left of Will and a $\frac{1}{2}$ probability that Jeremy is to the left of Nathan. These probabilities are independent, and since there are $4! = 24$ different permutations, our answer is just $24 \times \frac{1}{2} \times \frac{1}{2} = 6$.

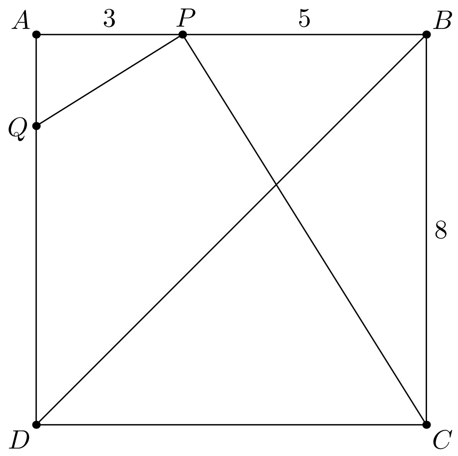
Gunga Bowl Set 3

B7. Let P be a point on side AB of square $ABCD$ with side length 8 such that $PA = 3$. Let Q be a point on side AD such that $PQ \perp PC$. The area of quadrilateral $PQDB$ can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{483}$

Solution: Consider the diagram below.



Since $\angle QAP = 90^\circ = \angle PBC$ and $\angle QPA = 90^\circ - \angle CPB = \angle PCB$, we get that $\triangle QAP \sim \triangle PBC$. By similarity ratios, we have $\frac{AQ}{AP} = \frac{BP}{BC} = \frac{5}{8}$, so $AQ = \frac{15}{8}$. Thus, the area of $\triangle APQ$ is $\frac{1}{2} \times 3 \times \frac{15}{8} = \frac{45}{16}$. Finally, the area of $PQDB$ is equal to the area of $\triangle ABD$ minus the area of $\triangle APQ$, which is just $32 - \frac{45}{16} = \frac{467}{16}$. The answer is 483.

B8. Jessica and Jeffrey each pick a number uniformly at random from the set $\{1, 2, 3, 4, 5\}$ (they could pick the same number). If Jessica's number is x and Jeffrey's number is y, the probability that x^y has a units digit of 1 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{31}$

Solution: Firstly, if $x = 1$, then x^y will always have a units digit of 1. The probability of this case is $\frac{1}{5}$. If $x = 2$ or $x = 4$, then x^y will never have a units digit of 1 because it is always even. Similarly, if $x = 5$, then x^y will always have a units digit of 5. Finally, when $x = 3$, x^y will only have a units digit of 1 if $y = 4$. The probability of this case is $\frac{1}{5} \times \frac{1}{5}$. Hence, our total probability is just $\frac{1}{5} + \frac{1}{5} \times \frac{1}{5} = \frac{6}{25}$. The answer is 31.

B9. For two points (x_1, y_1) and (x_2, y_2) in the plane, we define the *taxicab distance* between them as

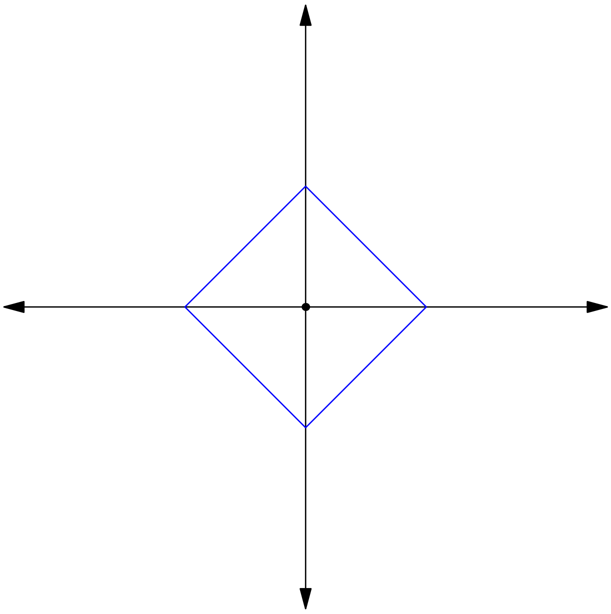
$$|x_1 - x_2| + |y_1 - y_2|.$$

For example, the taxicab distance between $(-1, 2)$ and $(3, \sqrt{2})$ is $6 - \sqrt{2}$. What is the largest number of points Nathan can find in the plane such that the taxicab distance between any two of the points is the same?

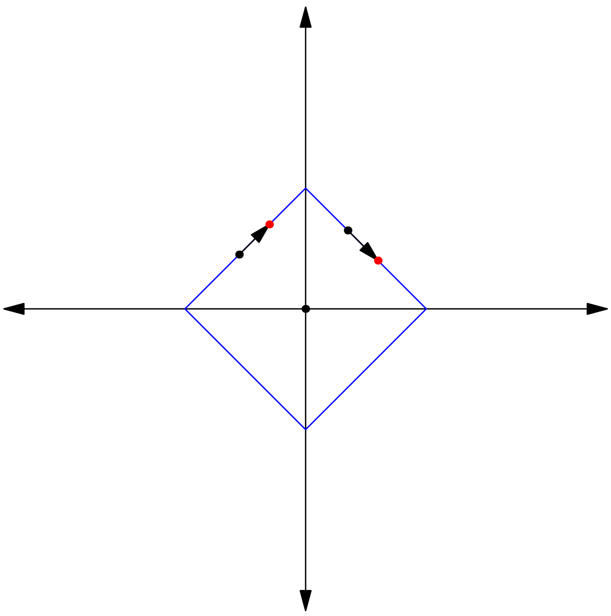
Proposed by: Nathan Xiong

Answer: $\boxed{4}$

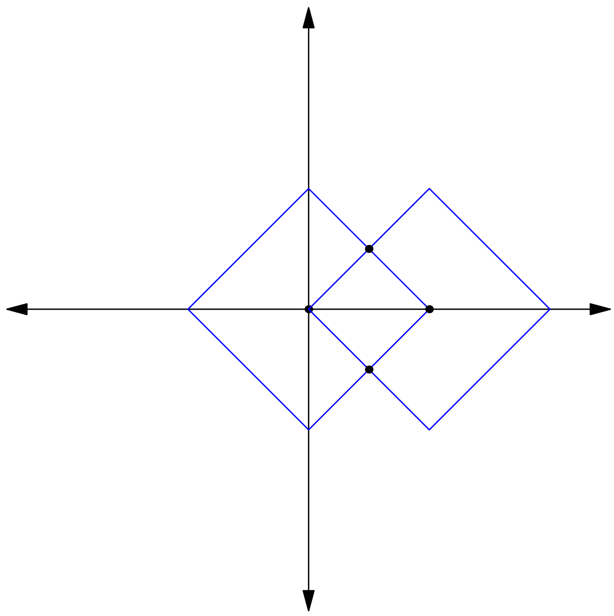
Solution 1: Assume without loss of generality that the common distance is 1. Translate the set of points such that one of them lands on the origin. Then, all other points in the set must lie on the unit circle, which happens to be the graph $|x| + |y| = 1$. It's easy to check that this is the graph of a diamond centered at the origin and with side length $\sqrt{2}$, as shown below.



Imagine "rotating" the points that are on the unit circle until one of them lands on a corner. This linear motion along the sides of the diamond preserves the "equilateralness" of our points, as long as we stop as soon as one of our points hits a corner. An example of this motion is shown below.



Assume without loss of generality that the point landing on the corner is $(1, 0)$. Then, all points other than $(1, 0)$ lie on the "circle" centered at $(1, 0)$, which is a diamond centered at $(1, 0)$ and with side length $\sqrt{2}$. These two diamonds intersect at exactly 2 points, and since all points other than $(0, 0)$ and $(1, 0)$ must lie on both of these diamonds, we can only have at most 4 points in our set.



Solution 2: For the curious reader, we present a much faster solution using more advanced tools.

We claim that the answer is 4. This is achievable with the points $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$. Now, we show that having more than 4 points is impossible.

Assume for the sake of contradiction that there exists a set of 5 or more points such that the taxicab distance between any two of the points is the same. Pick 5 of these points, and label them $(x_1,y_1),(x_2,y_2),\ldots,(x_5,y_5)$ such that $x_1 \le x_2 \le \cdots \le x_5$. By the Erdős-Szekeres theorem, there exist indices $1 \le i_1 < i_2 < i_3 \le 5$ such that either $y_{i_1} \le y_{i_2} \le y_{i_3}$ or $y_{i_1} \ge y_{i_2} \ge y_{i_3}$. Assume without loss of generality that $i_1 = 1$, $i_2 = 2$, and $i_3 = 3$ and that $y_{i_1} \le y_{i_2} \le y_{i_3}$. Then,

$$\begin{aligned}|x_1 - x_2| + |y_1 - y_2| + |x_2 - x_3| + |y_2 - y_3| &= x_2 - x_1 + x_3 - x_2 + y_2 - y_1 + y_3 - y_2 \\ &= x_3 - x_1 + y_3 - y_1 \\ &= |x_1 - x_3| + |y_1 - y_3|.\end{aligned}$$

Since the taxicab distances between (x_1,y_1), (x_2,y_2), and (x_3,y_3) should all be equal, we have a contradiction.

Gunga Bowl Set 4

B10. Will wants to insert some $\times$ symbols between the following numbers:

$$1 \quad 2 \quad 3 \quad 4 \quad 6$$

to see what kinds of answers he can get. For example, here is one way he can insert $\times$ symbols:

$$1 \times 23 \times 4 \times 6 = 552.$$

Will discovers that he can obtain the number 276. What is the sum of the numbers that he multiplied together to get 276?

Proposed by: Jeremy Zhou

Answer: $\boxed{52}$

Solution: Prime factorize 276 into $2^2 \times 3 \times 23$. Motivated by the fact that $2 \times 23 = 46$, let's group 4 and 6 together for right now. The remaining factorization is 2×3. Thus, Will can do

$$1 \times 2 \times 3 \times 46 = 276.$$

The answer is $1 + 2 + 3 + 46 = 52$.

B11. Let $ABCD$ be a parallelogram with $AB = 5$, $BC = 3$, and $\angle BAD = 60^\circ$. Let the angle bisector of $\angle ADC$ meet $\overline{AC}$ at E and $\overline{AB}$ at F. The length EF can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

Proposed by: Nathan Xiong

Answer: $\boxed{17}$

Solution: First, $\angle ADF = \frac{1}{2}\angle ADC = 60^\circ$. Hence, $\triangle ADF$ is equilateral and $AF = 3$. Clearly, $\triangle AEF \sim \triangle CED$, so $\frac{DE}{EF} = \frac{CD}{AF} = \frac{5}{3}$. Using the fact that $DE + EF = DF = 3$, we have $\frac{3-EF}{EF} = \frac{5}{3} \implies EF = \frac{9}{8}$. The answer is 17.

B12. Find the sum of all positive integers n such that $\lfloor \sqrt{n^2 - 2n + 19} \rfloor = n$.

Note: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Proposed by: Andrew Wen

Answer: $\boxed{35}$

Solution: By definition of the floor function, $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$ for all positive reals x. Therefore, if we set $x = \sqrt{n^2 - 2n + 19}$ we get

$$n \le \sqrt{n^2 - 2n + 19} < n + 1.$$

The left inequality implies that

$$n^2 \le n^2 - 2n + 19$$
$$n \le \frac{19}{2}.$$

The right inequality implies that

$$n^2 - 2n + 19 < n^2 + 2n + 1$$
$$\frac{9}{2} < n.$$

Hence, our possible solutions are 5, 6, 7, 8, and 9.

Lastly, we check that these do all work so our answer is $5 + 6 + 7 + 8 + 9 = 35$.

Gunga Bowl Set 5

B13. This year, February 29 fell on a Saturday. What is the next year in which February 29 will be a Saturday?

Proposed by: Nathan Xiong

Answer: $\boxed{2048}$

Solution: Clearly, our answer must be a leap year. Let's first find what day February 29 lies on in 2024. Since February 29, 2020 is on a Saturday, February 1, 2020 is also on a

Saturday. There are 366 days between February 1, 2020 and February 1, 2021, and since $366 \equiv 2 \pmod 7$, we get that February 1, 2021 is a Monday. Then, since $365 \equiv 1 \pmod 7$, February 1, 2024 is a Thursday, and February 29, 2024 is also a Thursday.

Thus, every four years, the day of February 29 moves forward 5 days (mod 7). Since 7 is prime, the day of February 29 will first cycle back to Saturday after $7 \times 4 = 28$ years. The answer is 2048.

B14. Let $f(x) = \frac{1}{x} - 1$. Evaluate

$$f\left(\frac{1}{2020}\right) \times f\left(\frac{2}{2020}\right) \times f\left(\frac{3}{2020}\right) \times \cdots \times f\left(\frac{2019}{2020}\right).$$

Proposed by: Nathan Xiong

Answer: $\boxed{1}$

Solution: The key observation is that $f(x) \cdot f(1-x) = 1$. Indeed,

$$f(x) \cdot f(1-x) = \left(\frac{1}{x} - 1\right)\left(\frac{1}{1-x} - 1\right) = \frac{1-x}{x} \cdot \frac{x}{1-x} = 1.$$

Thus, we can cancel out $f(\frac{1}{2020})$ with $f(\frac{2019}{2020})$, $f(\frac{2}{2020})$ with $f(\frac{2018}{2020})$, and so on. Everything collapses except for $f(\frac{1010}{2020})$, which also happens to equal 1.

B15. Square $WXYZ$ is inscribed in square $ABCD$ with side length 1 such that W is on AB, X is on BC, Y is on CD, and Z is on DA. Line WY hits AD and BC at points P and R respectively, and line XZ hits AB and CD at points Q and S respectively. If the area of $WXYZ$ is $\frac{13}{18}$, then the area of $PQRS$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. What is $m+n$?

Proposed by: Andrew Wen

Answer: $\boxed{21}$

Solution: Assume without loss of generality that $AW < BW$. By the Pythagorean theorem on $\triangle AWZ$, we have $AW^2 + AZ^2 = AW^2 + (1-AW)^2 = WZ^2 = \frac{13}{18}$. Solving this quadratic equation, we find $AW = \frac{1}{6}$ (we disregard the other solution because we assumed $AW < \frac{1}{2}$).

Let O be the center of $WXYZ$. By symmetry, $PQRS$ is a square, and O is also the center of $PQRS$. Since $\triangle RCY \sim \triangle RBW$, we have $5 = \frac{BW}{CY} = \frac{RW}{RY}$. We can manipulate

$$\frac{RW}{RY} = \frac{RY + 2OY}{RY} = 1 + \frac{2OY}{RY}.$$

Thus, $\frac{OY}{RY} = 2$ and $\frac{OY}{OR} = \frac{2}{3}$. Hence, the similarity ratio between $WXYZ$ and $PQRS$ is $2:3$. The area of $PQRS$ is then $\frac{9}{4} \times \frac{13}{18} = \frac{13}{8}$. The answer is 21.

Gunga Bowl Set 6

B16. Let ℓ_r denote the line $x + ry + r^2 = 420$. Jeffrey draws the lines ℓ_a and ℓ_b and calculates their single intersection point. Amazingly, he ends up with the point (a, b)! If a is a positive integer, what is $|b|$?

Proposed by: Nathan Xiong

Answer: $\boxed{14}$

Solution: We are given the two lines

$$\begin{cases} \ell_a: & x + ay + a^2 = 420, \\ \ell_b: & x + by + b^2 = 420. \end{cases}$$

The point (a, b) lies on both of these lines, so plugging it in gives

$$\begin{cases} a + ab + a^2 = 420, \\ a + b^2 + b^2 = 420. \end{cases}$$

Subtracting these two equations, we are left with $ab + a^2 = 2b^2$, which rearranges to $ab - b^2 = b^2 - a^2$. Since $a \neq b$, we can divide through by the common factor of $a - b$ to obtain $a = -2b$. Plugging this into the second equation above gives the quadratic $2b^2 - 2b - 420 = 0$, which has solutions $b = -14, 15$. Since a is positive, b must be negative, and the answer is $|-14| = 14$.

B17. Let set $\mathcal{L}$ consist of lines of the form $3x + 2ay = 60a + 48$ across all real constants a. For every line ℓ in $\mathcal{L}$, the point on ℓ closest to the origin is in set $\mathcal{T}$. The area enclosed by the locus of all the points in $\mathcal{T}$ can be expressed in the form $n\pi$ for some positive integer n. Compute n.

Proposed by: Andrew Wen

Answer: $\boxed{289}$

Solution: First, rearrange $3x + 2ay = 60a + 48$ into $3(x - 16) = 2a(30 - y)$. Note that the point $(16, 30)$ lies on every line in $\mathcal{L}$. Next, we can calculate the slope of this line to be $-\frac{3}{2a}$, so as a varies over all real numbers except 0, the slope of the line also varies over all real numbers except 0. Hence, we can characterize the set $\mathcal{L}$ by all lines passing through the point $(16, 30)$.

For each line $\ell \in \mathcal{L}$, its respective P is the foot from the origin to ℓ and thus must satisfy $\angle XPO = 90^\circ$, where O is the origin and $X = (8, 15)$. So, as ℓ varies in $\mathcal{L}$, the point P traces out $\mathcal{T}$, which is the circle with diameter OX. By the Pythagorean theorem, the radius of this circle is $\frac{1}{2}\sqrt{16^2 + 30^2} = 17$, and our answer is $17^2 = 289$.

B18. What is remainder when the 2020-digit number $202020 \cdots 20$ is divided by 275?

Proposed by: Jeffrey Shi

Answer: $\boxed{70}$

Solution: First, prime factorize 275 into $5^2 \times 11$.

Let N denote the 2020-digit number that we are studying. Note that we can write N as $20 + 20 \times 10^2 + 20 \times 10^4 + \cdots + 20 \times 10^{2018}$. We first evaluate N mod 25. Note that $10^n \equiv 0 \pmod{25}$ for all $n \geq 2$. Thus, all of the terms after 20 disappear under (mod 25), and we have $N \equiv 20 \pmod{25}$. Next, recall the divisibility rule for 11. The sum of the digits, starting from the right, with odd indices (units digit, hundreds digit, ...) is 0. The sum of the digits, starting from the right, with even indices (tens digit, thousands digit, ...) is $2 \times 1010 = 2020$. Thus, $N \equiv -2020 \equiv 4 \pmod{11}$.

Now, we have the two congruences

$$\begin{cases} N \equiv 20 \pmod{25}, \\ N \equiv 4 \pmod{11}. \end{cases}$$

The Chinese remainder theorem tells us that the unique solution is $N \equiv 70 \pmod{275}$, as desired.

Gunga Bowl Set 7

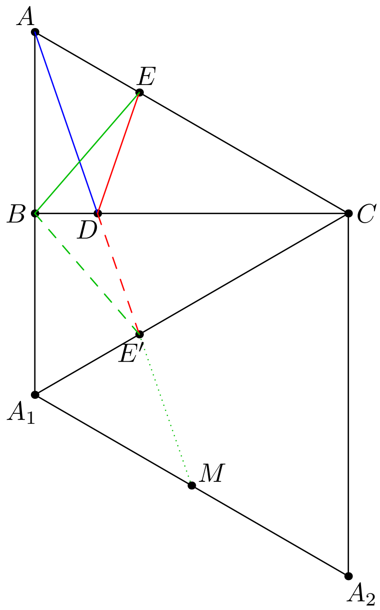
B19. Consider right triangle $\triangle ABC$ where $\angle ABC = 90^\circ$, $\angle ACB = 30^\circ$, and $AC = 10$. Suppose a beam of light is shot out from point A. It bounces off side BC and then bounces off side AC, and then hits point B and stops moving. If the beam of light travelled a distance of d, then compute d^2.

Proposed by: Andrew Wen

Answer: $\boxed{175}$

Solution:

Suppose the beam of light intersects segment BC at D and segment AC at E. Reflect A over BC to point A_1. Then, reflect A over A_1C to point A_2, and let M be the midpoint of segment A_1A_2. Since $\angle ACB = 30^\circ$, we know that $\triangle ACA_1$ is equilateral.



Imagine that our beam of light from A didn't reflect off of BC and instead travelled straight through BC. Suppose it intersects A_1C at E'. Clearly, E and E' are symmetric about BC. Next, since our beam of light was supposed to reflect off of E on AC and travel to B, by symmetry, our new beam of light should reflect off of A_1C and travel to B. If we again pretend that our beam of light travels straight through A_1C instead of reflecting, it will intersect A_1A_2 at a point M'. Clearly, B and M' are symmetric about A_1C. Thus, we actually have $M' = M$.

Now, we can calculate,

$$d = AD + DE + EB = AD + DE' + E'B = AD + DE' + E'M = AM.$$

We know that ACA_2A_1 is a rhombus with side length $2 \times 5 = 10$. The length of AM can be calculated easily using the Law of Cosines, but for the sake of avoiding trigonometry, we present another way to calculate it.

Let F be the foot from M to line AB. By the Pythagorean theorem, $BC = \sqrt{10^2 - 5^2} = 5\sqrt{3}$ and $MF = \frac{5}{2}\sqrt{3}$. By the Pythagorean theorem, $A_1F = \sqrt{5^2 - MF^2} = \frac{5}{2}$. One final application of the Pythagorean theorem gives $d^2 = AM^2 = AF^2 + MF^2 = (10 + \frac{5}{2})^2 + (\frac{5}{2}\sqrt{3})^2 = 175$.

B20. Let S be the set of all three digit numbers whose digits sum to 12. What is the sum of all the elements in S?

Proposed by: Nathan Xiong

Answer: $\boxed{31950}$

Solution: Let n be a number in S. There are several cases to consider.

- Case 1: n contains a 9.

  It's easy to check that there are 10 possible numbers here: 309, 390, 903, 930, 129, 192, 219, 291, 912, and 921. Their sum is 5196.

- Case 2: n starts with an 8.

  It's easy to check that there are 5 possible numbers here: 804, 813, 822, 831, 840. Their sum is 4110.

- Case 3: n neither starts with an 8 nor contains a 9.

  Write $n = \overline{abc}$ for digits a, b, c with $b, c \leq 8$, $a < 8$, and $a + b + c = 12$. We can consider the three digit number $\overline{a'b'c'}$ where $a' = 8 - a$, $b' = 8 - b$, and $c' = 8 - c$. Note that $a + b + c = 12$ if and only if $a' + b' + c' = 12$. Let X denote the subset of S consisting of the numbers less than 444 that don't contain a 9. Then, let Y denote the subset of S consisting of the numbers greater than 444 and less than 800 that don't contain a 9. Using this map from $\overline{abc}$ to $\overline{a'b'c'}$, we can pair every number in X with a number in Y and vice versa. We can also easily check that X and Y both have a cardinality of 25. Hence, the sum from this case is

$$\begin{aligned} 444 + \sum_{n\in X} n + \sum_{n \in Y} n &= 444 + \sum_{\overline{abc}\in X} \overline{abc} + \sum_{\overline{abc}\in X} \overline{a'b'c'} \\ &= 444 + \sum_{\overline{abc}\in X} (\overline{abc} + \overline{a'b'c'}) \\ &= 444 + \sum_{\overline{abc}\in X} 888 \\ &= 444 + 25 \times 888 \\ &= 22644 \end{aligned}$$

The answer is $5196 + 4110 + 22644 = 31950$.

B21. Consider all ordered pairs (m, n) where m is a positive integer and n is an integer that satisfy

$$m! = 3n^2 + 6n + 15,$$

where $m! = m \times (m - 1) \times \cdots \times 1$. Determine the product of all possible values of n.

Proposed by: Andy Xu

Answer: $\boxed{105}$

Solution: The key idea is to consider this equation (mod 7). We first check the cases for $m < 7$ manually.

- Case 1: $m = 1$.

  This quadratic has no integer solutions.
- Case 2: $m = 2$.

  This quadratic has no integer solutions.
- Case 3: $m = 3$.

  This quadratic has no integer solutions.
- Case 4: $m = 4$.

  This quadratic has solutions $n = -3, 1$.
- Case 5: $m = 5$.

  This quadratic has solutions $n = -7, 5$.
- Case 6: $m = 6$.

  This quadratic has no integer solutions.

Henceforth, assume $m \geq 7$, so $m! \equiv 0 \pmod{7}$. We take cases on the value of n mod 7.

- Case 1: $n \equiv 0 \pmod{7}$.

  Then, $3n^2 + 6n + 15 \equiv 1 \pmod{7}$, which is impossible.
- Case 2: $n \equiv 1 \pmod{7}$.

  Then, $3n^2 + 6n + 15 \equiv 3 \pmod{7}$, which is impossible.
- Case 3: $n \equiv 2 \pmod{7}$.

  Then, $3n^2 + 6n + 15 \equiv 4 \pmod{7}$, which is impossible.
- Case 4: $n \equiv 3 \pmod{7}$.

  Then, $3n^2 + 6n + 15 \equiv 4 \pmod{7}$, which is impossible.
- Case 5: $n \equiv 4 \pmod{7}$.

  Then, $3n^2 + 6n + 15 \equiv 3 \pmod{7}$, which is impossible.
- Case 6: $n \equiv 5 \pmod{7}$.

  Then, $3n^2 + 6n + 15 \equiv 1 \pmod{7}$, which is impossible.
- Case 7: $n \equiv 6 \pmod{7}$.

  Then, $3n^2 + 6n + 15 \equiv 5 \pmod{7}$, which is impossible.

Thus, there are no solutions for $m \geq 7$. The answer then is $(-3) \times 1 \times (-7) \times 5 = 105$.

Gunga Bowl Set 8

B22. Compute the number of ordered pairs of integers (m, n) satisfying $1000 > m > n > 0$ and

$$6 \cdot \operatorname{lcm}(m-n, m+n) = 5 \cdot \operatorname{lcm}(m, n).$$

Proposed by: Nathan Xiong

Answer: $\boxed{333}$

Solution: We first use the well-known fact $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$ to turn the equation into

$$6 \cdot \frac{m^2 - n^2}{\gcd(m-n, m+n)} = 5 \cdot \frac{mn}{\gcd(m, n)}.$$

Let $\nu_2(n)$ denote the largest integer e such that $2^e \mid n$. For example, $\nu_2(12) = 2$.

Lemma. We have

$$\gcd(m-n, m+n) = \begin{cases} 2\gcd(m,n) & \text{if } \nu_2(m) = \nu_2(n), \\ \gcd(m,n) & \text{otherwise.} \end{cases}$$

Proof. By the Euclidean algorithm, $\gcd(m-n, m+n) = \gcd(m+n, 2n)$. If $\nu_2(m) = \nu_2(n)$, then $\nu_2(m+n) \geq \nu_2(2n)$. Hence, we can bring out as many factors of 2 from $2n$ outside of the gcd as we want. In particular, $\gcd(m+n, 2n) = 2\gcd(m+n, n) = 2\gcd(m,n)$. If $\nu_2(m) \neq \nu_2(n)$, then $\nu_2(m+n) = \min\{\nu_2(m), \nu_2(n)\}$ and $\nu_2(m+n) < \nu_2(2n)$. Hence, we can delete one factor of 2 from $2n$, so $\gcd(m+n, 2n) = \gcd(m+n, n) = \gcd(m,n)$. This proves the lemma. □

Using this lemma, we can split the problem into two cases.

- Case 1: $\nu_2(m) = \nu_2(n)$.

  Using the lemma, the equation simplifies to

  $$3(m^2 - n^2) = 5mn \implies \frac{m}{n} - \frac{n}{m} = \frac{5}{3}.$$

  Letting $x = \frac{m}{n}$ and solving the resulting quadratic, we find $x = \frac{5+\sqrt{61}}{6}$, so this case is impossible.

- Case 2: $\nu_2(m) \neq \nu_2(n)$.

  Using the lemma, the equation simplifies to

  $$6(m^2 - n^2) = 5mn \implies \frac{m}{n} - \frac{n}{m} = \frac{5}{6}.$$

  Letting $x = \frac{m}{n}$ and solving the resulting quadratic, we find $x = \frac{3}{2}$. Thus, let $m = 3k$ and $n = 2k$ for a positive integer k. Note that this will always satisfy the condition $\nu_2(m) \neq \nu_2(n)$. Plugging this into our equation, we see that

  $$6(m^2 - n^2) = 6(9k^2 - 4k^2) = 30k^2 = 5 \cdot 3k \cdot 2k = 5mn,$$

  so all pairs (m, n) of the form $(3k, 2k)$ work. Since $m = 3k < 1000$, we have $1 \leq k \leq 333$, and there are 333 working pairs in this case.

The answer is 333.

B23. Andrew is flipping a coin ten times. After every flip, he records the result (heads or tails). He notices that after every flip, the number of heads he had flipped was always at least the number of tails he had flipped. In how many ways could Andrew have flipped the coin?

Proposed by: Nathan Xiong

Answer: 252

Solution: We first restate our problem in the Cartesian plane. Consider an ant at the origin of the plane. Whenever Andrew flips heads, the ant moves one unit in the positive x direction. Whenever Andrew flips tails, the ant moves one unit in the positive y direction. The ant will trace out a path of length 10. The condition that the number of heads flipped is always at least the number of tails flipped is equivalent to the ant's path never crossing the line $y = x$. Here's an example of a legal path:

INSERT ASY

We start with some definitions for our convenience. In general, we will refer to a *path* as a sequence of right and up moves (always 1 unit at a time) starting at the origin in the Cartesian plane. Call the line $y = x + 1$ the *fatal line*. Note that if a path crosses the line $y = x$, it must intersect the fatal line. If a path intersects the fatal line, call it a *bad* path.

The first case we consider is when Andrew flips five heads and five tails. In this case, our path ends on $(5, 5)$. We will use complementary counting and count the number of bad paths going to $(5, 5)$. There are $\binom{10}{5}$ total paths from $(0, 0)$ to $(5, 5)$. Now, consider any bad path. Reflect the part of the path after the intersection point with the fatal line about the fatal line. Here is an example:

INSERT ASY

Initially, our path had 5 right moves and 5 up moves. When we intersect the fatal line, we've moved right i times and up $i + 1$ times for some i. We have $5 - i$ remaining right moves and $4 - i$ remaining up moves. After we reflect part of the path however, these switch, and we have $4-i$ right moves and $5-i$ up moves. Hence, in our modified path, the total number of right moves is 4 and the total number of up moves is 6. So, our modified path will go to $(4, 6)$. Using this reflection trick, it's easy to see that there is a bijection between bad paths in this case and paths going to $(4, 6)$. Every path going to $(4, 6)$ must intersect the fatal line, so we can just reflect the part of the path after that intersection point to recover the bad path going to $(5, 5)$. Since there are $\binom{10}{4}$ paths going to $(4, 6)$, the number of valid paths in our first case is $\binom{10}{5} - \binom{10}{4}$.

The next case is when we flip six heads and four tails. We can use the same argument above. There is a bijection between bad paths in this case and paths going to $(3, 7)$. Hence, the number of valid paths in this case is $\binom{10}{4} - \binom{10}{3}$.

We can continue this argument all the way to ten heads and zero tails. The answer is then $\left(\binom{10}{5} - \binom{10}{4}\right) + \left(\binom{10}{4} - \binom{10}{3}\right) + \left(\binom{10}{3} - \binom{10}{2}\right) + \left(\binom{10}{2} - \binom{10}{1}\right) + \left(\binom{10}{1} - \binom{10}{0}\right) + \binom{10}{0} = \binom{10}{5} = 252$.

Remark: This reflection trick is commonly used in these sorts of block walking problems. It can be used to prove the formula for the Catalan numbers.

B24. Consider a triangle ABC with $AB = 7$, $BC = 8$, and $CA = 9$. Let D lie on $\overline{AB}$ and E lie on $\overline{AC}$ such that $BCED$ is a cyclic quadrilateral and D, O, E are collinear, where O is the circumcenter of ABC. The area of $\triangle ADE$ can be expressed as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. What is $m + n + p$?

Proposed by: Nathan Xiong

Answer: $\boxed{177}$

Solution: Firstly, since $BCED$ is cyclic, easy angle chasing gives us $\triangle AED \sim \triangle ABC$. By Heron's formula, the area of $\triangle ABC$ is $12\sqrt{5}$. It suffices to find the ratio of similitude between $\triangle AED$ and $\triangle ABC$.

The key claim is that $\overline{AO} \perp \overline{DE}$. This isn't difficult to prove using angle chasing. Since $\triangle AOC$ is an isosceles triangle and $\angle AOC = 2\angle ABC$, we know that $\angle OAE = 90^\circ - \angle ABC$. By our similar triangles, we also know that $\angle AEO = \angle ABC$. Hence, $\angle AOE = 180^\circ - (90^\circ - \angle ABC) - \angle ABC = 90^\circ$, as desired.

Now, let X be the foot from A to $\overline{BC}$. Since $\overline{AO}$ and $\overline{AX}$ are the altitudes of $\triangle AED$ and $\triangle ABC$ respectively, the ratio of similitude between the two triangles is $\frac{AO}{AX}$. By the circumradius area formula, we have $12\sqrt{5} = \frac{7\cdot 8\cdot 9}{4R}$, so $AO = R = \frac{21\sqrt{5}}{10}$. It's also easy to see that $AX = \frac{2\cdot 12\sqrt{5}}{8} = 3\sqrt{5}$. Our desired ratio is then $\frac{21\sqrt{5}/10}{3\sqrt{5}} = \frac{7}{10}$.

The area of $\triangle ADE$ is $(\frac{7}{10})^2 \times 12\sqrt{5} = \frac{147\sqrt{5}}{25}$. The answer is $147 + 5 + 25 = 177$.

Gunga Bowl Set 9

This set consists of three estimation problems, with scoring schemes described.

B25. Submit one of the following ten numbers:

$$3 \qquad 6 \qquad 9 \qquad 12 \qquad 15 \qquad 18 \qquad 21 \qquad 24 \qquad 27 \qquad 30.$$

The number of points you will receive for this question is equal to the number you selected divided by the total number of teams that selected that number, then rounded up to the nearest integer. For example, if you and four other teams select the number 27, you would receive $\lceil \frac{27}{5} \rceil = 6$ points.

Proposed by: William Yue

B26. Submit any integer from 1 to 1,000,000, inclusive. The *standard deviation* σ of all responses x_i to this question is computed by first taking the arithmetic mean μ of all responses, then taking the square root of average of $(x_i - \mu)^2$ over all i. More, precisely, if there are N responses, then

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2}.$$

For this problem, your goal is to estimate the standard deviation of all responses.

An estimate of e gives $\max\left\{\left\lceil 130 \cdot \left(\min\left\{\frac{\sigma}{e}, \frac{e}{\sigma}\right\}\right)^3\right\rceil - 100, 0\right\}$ points.

Proposed by: William Yue

B27. For a positive integer n, let $f(n)$ denote the number of distinct nonzero exponents in the prime factorization of n. For example, $f(36) = f(2^2 \times 3^2) = 1$ and $f(72) = f(2^3 \times 3^2) = 2$. Estimate

$$N = f(2) + f(3) + \cdots + f(10000).$$

An estimate of e gives $\max\{30 - \lfloor 7\log_{10}(|N - e|)\rfloor, 0\}$ points.

Proposed by: Nathan Xiong

Answer: $\boxed{14031}$

Remark: This is OEIS sequence A071625.