MOAA 2020: Theme Round Solutions

October 10th, 2020

Optimization

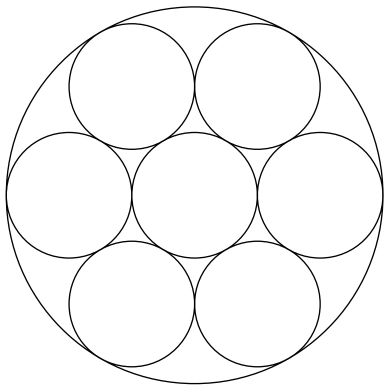
Mathematical **optimization** is all about finding the maximum or minimum of a value under a given set of conditions. Try your hand on several optimization problems below!

TO1. What is the maximum number of circles of radius 1 you can fit, without overlapping them, in a circle of radius 3?

Proposed by: William Yue

Answer: $\boxed{7}$

Solution: The answer is 7, achievable by the construction shown below, which is clearly optimal.



TO2. The Den has two deals on chicken wings. The first deal is 4 chicken wings for 3 dollars, and the second deal is 11 chicken wings for 8 dollars. If Jeremy has 18 dollars, what is the largest number of chicken wings he can buy?

Proposed by: Nathan Xiong

Answer: $\boxed{24}$

Solution: Suppose Jeremy uses the first deal a times and the second deal b times. Then, he spends $3a+8b \leq 18$ dollars to get $4a+11b$ chicken wings. Let's take cases on the value of b:

- Case 1: $b = 0$.

  Then, we need $3a \leq 18$, so $a = 6$ is the maximum. This gives $4 \times 6 = 24$ chicken wings.

- Case 2: $b = 1$.

  Then, we need $3a \leq 10$, so $a = 3$ is the maximum. This gives $4 \times 3 + 11 \times 1 = 23$ chicken wings.

- Case 3: $b = 2$.

  Then, we need $3a \leq 2$, so $a = 0$ is the maximum. This gives $11 \times 2 = 22$ chicken wings.

Jeremy cannot use the third deal more than two times since he doesn't have enough money. Hence, the largest number of chicken wings Jeremy can buy is 24.

TO3. Consider the addition

$$\begin{array}{ccccc} & & \mathrm{O} & \mathrm{N} & \mathrm{E} \\ + & & \mathrm{T} & \mathrm{W} & \mathrm{O} \\ \hline & \mathrm{F} & \mathrm{O} & \mathrm{U} & \mathrm{R} \end{array}$$

where different letters represent different **nonzero** digits. What is the smallest possible value of the four-digit number FOUR?

Proposed by: Nathan Xiong

Answer: $\boxed{1236}$

Solution: Since we are adding two three-digit numbers to get a four-digit number, we know that $\mathrm{F} = 1$. Now, if we wish to minimize FOUR, we want to take $\mathrm{O} = 2$. Let's see if this is possible; the addition has become

$$\begin{array}{ccccc} & & 2 & \mathrm{N} & \mathrm{E} \\ + & & \mathrm{T} & \mathrm{W} & 2 \\ \hline & 1 & 2 & \mathrm{U} & \mathrm{R} \end{array}$$

Since we know that $\mathrm{T} \neq 0$, in order for the hundreds digits to match we need $\mathrm{T} = 9$ and the sum $\mathrm{N} + \mathrm{W}$ to carry over an extra 1. In addition, to continue to minimize FOUR we want $\mathrm{U} = 3$. Adding in this information gives

$$\begin{array}{ccccc} & & 2 & \mathrm{N} & \mathrm{E} \\ + & & 9 & \mathrm{W} & 2 \\ \hline & 1 & 2 & 3 & \mathrm{R} \end{array}$$

It now suffices to find the minimum possible value for R. Since $\mathrm{E} \neq 9$ and if $\mathrm{E} = 8$ then $\mathrm{R} = 0$ which isn't allowed, we know that $\mathrm{E} + 2 = \mathrm{R}$. Since the digits $1, 2, 3$ are already used, this means that $\mathrm{E} = 4$ and $\mathrm{R} = 6$ is the smallest possibility. We can confirm that this indeed works if we take $\mathrm{N} = 5$ and $\mathrm{W} = 8$, for example, resulting in the final addition of

$$\begin{array}{ccccc} & & 2 & 5 & 4 \\ + & & 9 & 8 & 2 \\ \hline & 1 & 2 & 3 & 6 \end{array}$$

Therefore, 1236 is the minimum possible value for FOUR.

TO4. Over all real numbers x, let k be the minimum possible value of the expression

$$\sqrt{x^2 + 9} + \sqrt{x^2 - 6x + 45}.$$

Determine k^2.

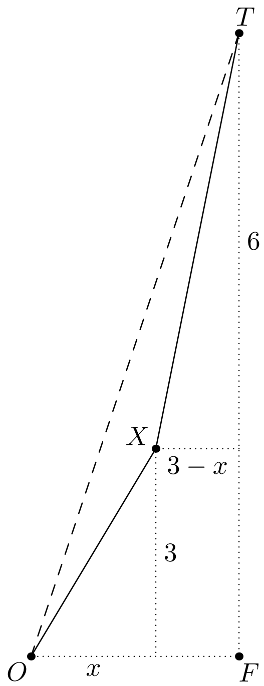
Proposed by: Jeffrey Shi

Answer: $\boxed{90}$

Solution: The key is to rewrite the expression by completing the square:

$$\sqrt{x^2 + 9} + \sqrt{x^2 - 6x + 45} = \sqrt{x^2 + 9} + \sqrt{(3 - x)^2 + 36}.$$

Now, we can interpret this geometrically using the Pythagorean theorem! For any real number x, the first quantity represents the distance from the point $(0,0)$ to the point $(x,3)$, while the second quantity represents the distance from the point $(x,3)$ to the point $(3,9)$, as shown below:



Therefore, the expression in the problem is equal to the length of OX plus the length of XT. By the triangle inequality, this is minimized when X lies on OT, and our answer is just the length of OT. The Pythagorean theorem tells us that this length is

$$k = \sqrt{3^2 + 9^2} = \sqrt{90},$$

so $k^2 = 90$.

TO5. For a real number x, the minimum value of the expression

$$\frac{2x^2 + x - 3}{x^2 - 2x + 3}$$

can be written in the form $\frac{a - \sqrt{b}}{c}$, where a, b, c are positive integers such that b is not divisible by the square of any prime. Find $a + b + c$.

Proposed by: Jeffrey Shi

Answer: $\boxed{74}$

Solution: Let the expression equal k. We can then rearrange to find

$$2x^2 + x - 3 = k(x^2 - 2x + 3)$$
$$(2 - k)x^2 - (2k + 1)x - 3k - 3 = 0$$

We know that since x is real, the discriminant of this quadratic must be greater than or equal to zero.

$$\begin{aligned}(2k+1)^2 - 4(2-k)(-3k-3) &\geq 0\\ -8k^2 + 16k + 25 &\geq 0\\ 8k^2 - 16k - 25 &\leq 0.\end{aligned}$$

Remember that we are trying to minimize k, and from this quadratic inequality we find

$$\frac{4-\sqrt{66}}{4} \leq k \leq \frac{4+\sqrt{66}}{4}.$$

Thus the minimum value of $k = \frac{4-\sqrt{66}}{4}$, and our answer is 74. Note that this value of k is clearly achievable since the resulting quadratic equation has real roots.

Errata: There was a minor error in the problem statement, which stated that a and c were relatively prime.

Relay

Each problem in this section will depend on the previous one! The values $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ refer to the answers to problems 1, 2, 3, and 4, respectively.

TR1. The number 2020 has three different prime factors. What is their sum?

Proposed by: William Yue

Answer: $\boxed{108}$

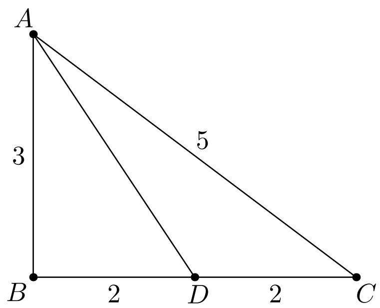
Solution: We can factor $2020 = 2^2 \times 5 \times 101$. The three different prime factors are 2, 5, and 101, and their sum is $2 + 5 + 101 = 108$.

TR2. Let $\mathcal{A}$ be the answer to the previous problem. Suppose ABC is a triangle with $AB = 81$, $BC = \mathcal{A}$, and $\angle ABC = 90°$. Let D be the midpoint of $\overline{BC}$. The perimeter of $\triangle CAD$ can be written as $x + y\sqrt{z}$, where x, y, and z are positive integers and z is not divisible by the square of any prime. What is $x + y$?

Proposed by: Nathan Xiong

Answer: $\boxed{216}$

Solution: Since $\mathcal{A} = 108$, we know that $BC = 108$. Now, we recognize that $108 = 27 \times 4$ and $81 = 27 \times 3$, so ABC is a $3-4-5$ right triangle, with hypotenuse $27 \times 5 = 135$. Let's solve the scaled down version of the problem, where $AB = 3, BC = 4, CA = 5$, then scale our final answer back up by 27.



To compute the perimeter of $\triangle CAD$, we need to compute AD using the Pythagorean theorem: $AD = \sqrt{3^2 + 2^2} = \sqrt{13}$. Thus, the perimeter is $7 + \sqrt{13}$. Scaling back up gives $189 + 27\sqrt{13}$, so the answer is $x + y = 189 + 27 = 216$.

TR3. Let $\mathcal{B}$ the answer to the previous problem. What is the unique real value of k such that the parabola $y = \mathcal{B}x^2 + k$ and the line $y = kx + \mathcal{B}$ are tangent?

Proposed by: Andrew Wen

Answer: $\boxed{432}$

Solution: We wish to solve the equation $\mathcal{B}x^2 + k = kx + \mathcal{B}$ and only find one solution, as this would imply that the line is tangent to the parabola. This means that the quadratic $\mathcal{B}x^2 - kx + (k - \mathcal{B})$ must have a double root, which means that the discriminant $\Delta = b^2 - 4ac$ must be zero:

$$0 = \Delta = k^2 - 4\mathcal{B}(k - \mathcal{B}) = k^2 - 4\mathcal{B}k + 4\mathcal{B}^2 = (k - 2\mathcal{B})^2.$$

Hence, $k = 2\mathcal{B} = 2 \times 216 = 432$.

TR4. Let $\mathcal{C}$ be the answer to the previous problem. How many ordered triples of positive integers (a, b, c) are there such that $\gcd(a, b) = \gcd(b, c) = 1$ and $abc = \mathcal{C}$?

Proposed by: Andrew Wen

Answer: $\boxed{30}$

Solution: Prime factorize $\mathcal{C} = 432 = 2^4 \times 3^3$. Now, we can write $a = 2^{a_1}3^{a_2}$, $b = 2^{b_1}3^{b_2}$, and $c = 2^{c_1}3^{c_2}$, where $a_1+b_1+c_1 = 4$ and $a_2+b_2+c_2 = 3$. However, by the gcd condition, we also know that

$$\min\{a_1, b_1\} = \min\{a_2, b_2\} = \min\{b_1, c_1\} = \min\{b_2, c_2\} = 0.$$

Hence,

- For the triple (a_1, b_1, c_1), we know that either $b_1 = 0$ or $a_1 = c_1 = 0$. In the first case, we are left with $a_1 + c_1 = 4$, which gives 5 possibilities, while in the second case, we are left with $b_1 = 4$, which is 1 possibility. Thus, there are 6 possibilities for the triple (a_1, b_1, c_1).
- For the triple (a_2, b_2, c_2), we also know that either $b_2 = 0$ or $a_2 = c_2 = 0$. In the first case, we are left with $a_1 + c_1 = 3$, which gives 4 possibilities, while in the second case, we are left with $b_2 = 3$, which is 1 possibility. Thus, there are 5 possibilities for the triple (a_2, b_2, c_2).

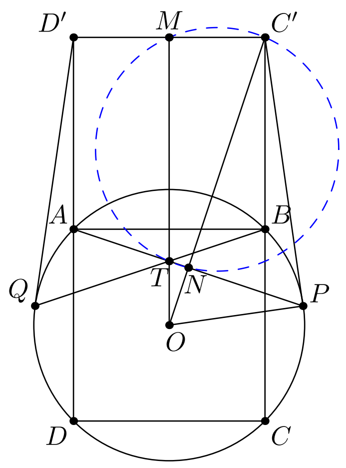
In total, this gives $6 \times 5 = 30$ possibilities for (a, b, c).

TR5. Let $\mathcal{D}$ be the answer to the previous problem. Let $ABCD$ be a square with side length $\mathcal{D}$ and circumcircle ω. Denote points C' and D' as the reflections over line AB of C and D respectively. Let P and Q be the points on ω, with A and P on opposite sides of line BC and B and Q on opposite sides of line AD, such that lines $C'P$ and $D'Q$ are both tangent to ω. If the lines AP and BQ intersect at T, what is the area of $\triangle CDT$?

Proposed by: Andrew Wen

Answer: $\boxed{375}$

Solution: Let O be the center of ω. Note that $\angle OAC' = \angle OAB + \angle BAC' = 45° + 45° = 90°$, so $C'A$ is tangent to ω. This also implies that $AP \perp OC'$. Let M and N be the feet from O to $C'D'$ and AP respectively. Since $\angle TNC' = \angle TMC' = 90°$, we know that $TNC'M$ is cyclic.



Therefore, we can apply power of a point at O: $ON \cdot OC' = OT \cdot OM$. However, since $\triangle ONP \sim \triangle OPC'$ because $\angle NOP = \angle POC'$ and $\angle ONP = 90° = \angle OPC'$, we know that $\frac{ON}{OP} = \frac{OP}{OC'} \implies ON \cdot OC' = OP^2$. Now we can finish the problem. Since $\mathcal{D} = 30$, we know that $OM = 15 + 30 = 45$ and $OP = OA = 15\sqrt{2}$. Therefore,

$$45 \cdot OT = OM \cdot OT = OP^2 = 450 \implies OT = 10.$$

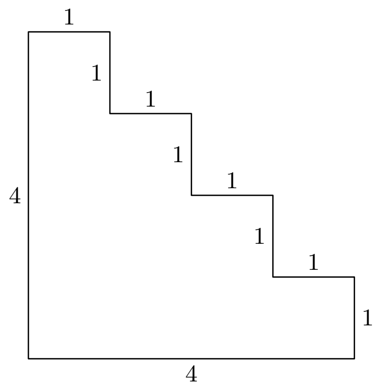
Therefore, the area of $\triangle CDT$ is $\frac{1}{2} \times 30 \times (15 + 10) = 375$.

to Bash or not to Bash...

In math competitions, we call a problem "bashy" if it involves a lot of routine yet annoying computations and/or casework. The following problems all seem bashy at first glance, but we promise they all have nicer solutions that don't require too much computation.

Do you have the ingenuity to find the fast and clever approaches? Or would you prefer to spend the extra time and effort to simply bash out the answer? To bash or not to bash, that is your question to answer.

TB1. Find the perimeter of the figure below, where all angles are right angles.



Proposed by: Nathan Xiong

Answer: $\boxed{16}$

Solution: The trick is to note that the total horizontal length is $2 \times 4 = 8$, and the total vertical length is also $2 \times 4 = 8$, so the perimeter is just $8 + 8 = 16$.

TB2. Evaluate

$$1 \times 5 + 2 \times 8 + 3 \times 13 + 5 \times 21 + 8 \times 34 + 13 \times 55.$$

Proposed by: Nathan Xiong

Answer: $\boxed{1152}$

Solution: While this problem looks like a large computation, there's actually a slick solution using difference of squares. We actually apply this relation in the reverse direction to find that

$$\begin{aligned}
&1 \times 5 + 2 \times 8 + 3 \times 13 + 5 \times 21 + 8 \times 34 + 13 \times 55 \\
&\quad = (3-2)(3+2) + (5-3)(5+3) + (8-5)(8+5) \\
&\qquad + (13-8)(13+8) + (21-13)(21+13) + (34-21)(34+21) \\
&\quad = (3^2 - 2^2) + (5^2 - 3^2) + (8^2 - 5^2) + (13^2 - 8^2) + (21^2 - 13^2) + (34^2 - 21^2) \\
&\quad = 34^2 - 2^2 \\
&\quad = 1152.
\end{aligned}$$

TB3. Jeff and Geoff each choose a random number from the set $\{1, 2, \ldots, 100\}$. Let Jeff's number be a and Geoff's number be b. Given that $a \neq b$, the probability that $ab + a + b$ is odd can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

Proposed by: Andrew Wen

Answer: $\boxed{347}$

Solution: The key is to recognize that $ab + a + b = (a + 1)(b + 1) - 1$. Let's employ complementary counting and instead compute the probability that this expression is even. This is equal to the probability that $(a + 1)(b + 1)$ is odd, which only occurs when both factors $a+1$ and $b+1$ are odd. The probability of this occurring is equal to the probability that both a and b are even, which occurs with chance $\frac{50}{100} \cdot \frac{49}{99} = \frac{49}{198}$. Therefore, our desired probability is $1 - \frac{49}{198} = \frac{149}{198}$, which gives the answer of $149 + 198 = 347$.

TB4. Find the number of positive integers n less than 100 that satisfy the equation

$$n = \lfloor \sqrt{n} \rfloor \cdot \lceil \sqrt{n} \rceil.$$

Note: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the least integer greater than or equal to x.

Proposed by: Nathan Xiong

Answer: $\boxed{18}$

Solution: Let $\lfloor \sqrt{n} \rfloor = k$. Then,

- If $n = k^2$ is a perfect square, then $\lceil \sqrt{n} \rceil = k$ as well, and the equation is satisfied.
- If $(k + 1)^2 > n > k^2$ is not a perfect square, then $\lceil \sqrt{n} \rceil = k + 1$, so the only valid value of n for the equation to be satisfied is $k(k + 1)$.

Therefore, we need to find the number of positive integers n less than 100 that are of the form k^2 or $k(k + 1)$. There are 9 perfect squares less than 100, and each of them also has an associated $k(k + 1)$ term, resulting in 18 total possibilities.

TB5. Arnav writes every positive integer factor of 2020^2 exactly once on a blackboard. Every minute, he chooses a number on the blackboard uniformly at random, and he erases it as well as all of its factors. The expected amount of minutes that Arnav takes to erase every number on the board can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by: William Yue

Answer: $\boxed{18737}$

Solution: Call a number *selected* if it is a number which Arnav picks to erase, along with its factors (in particular, not all erased numbers are selected). Now, the key is to notice that the number of minutes it takes Arnav to erase all the numbers is equal to the total number of selected numbers.

If $\mathcal{F}$ is the set of factors of $2020^2 = 2^4 \cdot 5^2 \cdot 101^2$, then we can define indicator variables X_i, for each $i \in \mathcal{F}$, to be 1 if the number is selected and 0 otherwise. Then, we are seeking the quantity

$$\mathbb{E}\left[\sum_{i \in \mathcal{F}} X_i\right] = \sum_{i \in \mathcal{F}} \mathbb{E}[X_i]$$

by Linearity of Expectation. However, note that for any individual factor i of 2020^2, the probability that it is selected is equal to 1 over its number of multiples which are also in

$\mathcal{F}$ (this is because i is erased whenever it or one of its other multiples is selected, and each of these events occurs with equal probability). Thus, we can write this sum as

$$\begin{aligned}
\sum_{a=0}^{4}\sum_{b=0}^{2}\sum_{c=0}^{2}\mathbb{E}[X_{2^{4-a}.5^{2-b}.101^{2-c}}] &= \sum_{a=0}^{4}\sum_{b=0}^{2}\sum_{c=0}^{2}\frac{1}{(a+1)(b+1)(c+1)} \\
&= \left(\sum_{a=0}^{4}\frac{1}{a+1}\right)\left(\sum_{b=0}^{2}\frac{1}{b+1}\right)\left(\sum_{c=0}^{2}\frac{1}{c+1}\right) \\
&= \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)\left(1+\frac{1}{2}+\frac{1}{3}\right)^2 \\
&= \frac{16577}{2160}.
\end{aligned}$$

The answer is $16577 + 2160 = 18737$.

Functions

The following problems all involve functions. A function can be thought of as a process that takes in some input and spits out some output. For example, $f(x) = x^2 - 1$ is a simple function which takes in 3 and spits out $f(3) = 3^2 - 1 = 8$.

TF1. We define the *prime-counting function* $\pi(n)$ as the number of primes that are less than or equal to n. For example, $\pi(8) = 4$ since there are 4 primes less than or equal to 8, namely, 2, 3, 5, and 7. What is the sum of all positive integers n with $\pi(n) = 6$?

Note: The π used here is unrelated to the infinite decimal $3.1415\ldots$

Proposed by: Nathan Xiong

Answer: $\boxed{58}$

Solution: We need to find all positive integers n such that there are exactly 6 prime numbers less than or equal to n. The first seven primes are $2, 3, 5, 7, 11, 13, 17$, so the first number n such that $\pi(n) = 6$ is $n = 13$, and the first number for which $\pi(n) = 7$ is $n = 17$. Therefore, the only numbers n for which $\pi(n) = 6$ are 13, 14, 15, and 16. The answer is 58.

TF2. Consider the function $f(x) = 2020 - x$. Find $f(f(f(f(f(1)))))$.

Proposed by: Nathan Xiong

Answer: $\boxed{2019}$

Solution: The key is to notice that $f(f(x)) = 2019 - (2019 - x) = x$. Therefore, $f(f(f(f(f(1))))) = f(f(f(1))) = f(1) = 2019$.

TF3. Consider the polynomial $P(x)$ such that for any positive real number x, $P(x)$ equals the numerical sum of the volume and surface area of regular tetrahedron with side length x. If r is the only nonzero real root of this polynomial, determine r^2.

Proposed by: William Yue

Answer: $\boxed{216}$

Solution: Since the volume of a regular tetrahedron is proportional to the cube of its side length, if V is the volume of a regular unit tetrahedron (side length 1), then the volume of a regular tetrahedron with side length x is Vx^3. Similarly, since the surface area of a regular tetrahedron is proportional the square of its side length, if A is the surface area of a regular unit tetrahedron, then the surface area of a regular tetrahedron with side length x is Ax^2. Therefore, the relevant polynomial is

$$P(x) = Vx^3 + Ax^2 = x^2(Vx + A),$$

whose only nonzero root is at $r = -\frac{A}{V}$. We calculate A and V separately.

- First, we find the surface area of a regular unit tetrahedron. This area is composed of four equilateral triangles with side length 1. The area of each triangle is $\frac{\sqrt{3}}{4} \times 1^2$, so the surface area is $A = 4 \times \frac{\sqrt{3}}{4} = \sqrt{3}$.
- Next, we find the volume of a regular unit tetrahedron. Label its four vertices A, B, C, and D. Drop the altitude from a vertex A to the center M of the opposite equilateral triangle face BCD. We apply the Pythagorean Theorem on right triangle AMB. We know that $AB = 1$ and $MB = \frac{2}{3} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3}$. Hence,

$$AM = \sqrt{AB^2 - MB^2} = \sqrt{1 - \frac{1}{3}} = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}.$$

Therefore, the volume is

$$V = \frac{1}{3} \cdot \frac{\sqrt{3}}{4} \cdot \frac{\sqrt{6}}{3} = \frac{\sqrt{2}}{12}.$$

Putting these two together, we find that the only nonzero root of $P(x)$ is $r = -\frac{\sqrt{3}}{\sqrt{2}/12} = -\frac{12\sqrt{3}}{\sqrt{2}}$. The answer is $r^2 = \frac{144 \cdot 3}{2} = 216$.

TF4. Let $P(x) = a_5x^5 + a_4x^4 + \cdots + a_1x + a_0$ be the unique polynomial of degree 5 satisfying

$$\begin{aligned} P(0) &= 1, \\ P(1) &= 2, \\ P(2) &= 4, \\ P(3) &= 8, \\ P(4) &= 16, \\ P(5) &= 32. \end{aligned}$$

Given that there is a unique positive integer n such that $P(n) = 1024$, find n.

Proposed by: Nathan Xiong

Answer: $\boxed{11}$

Solution: The trick is to define the polynomials

$$\binom{x}{k} = \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}$$

as the standard binomial coefficient functions with extended domains to all reals. Note that when x is a non-negative integer less than k, $\binom{x}{k} = 0$. Thus,

$$P(x) = \binom{x}{0} + \binom{x}{1} + \binom{x}{2} + \binom{x}{3} + \binom{x}{4} + \binom{x}{5}$$

satisfies all the conditions in the problem and hence is our unique polynomial. Finally, using the fact that $\binom{n}{k} = \binom{n}{n-k}$, check that

$$P(11) = \binom{11}{0} + \binom{11}{1} + \binom{11}{2} + \binom{11}{3} + \binom{11}{4} + \binom{11}{5} = \frac{1}{2}\sum_{i=0}^{11}\binom{11}{i} = 2^{10} = 1024.$$

So, the unique value of n such that $P(n) = 1024$ is $n = 11$.

TF5. Determine the number of bijections f on the set $\{1, 2, \ldots, 8\}$ satisfying

$$|2f(i) - 2i - 1| \leq 3,$$

for all $i = 1, 2, \ldots, 8$.

Note: A bijection on a finite set X is a function from X to itself such that for any $a, b \in X$ with $a \neq b$, then $f(a) \neq f(b)$.

Proposed by: Andrew Wen

Answer: $\boxed{81}$

Solution: First, we decipher what the condition means:

$$|2f(i) - 2i - 1| \leq 3 \implies -2 \leq 2(f(i) - i) \leq 4 \implies -1 \leq f(i) - i \leq 2.$$

We will solve the problem using recursion. Let a_n denote the number of bijections on $\{1, 2, \ldots, n\}$ satisfying the problem's condition. One can check that $a_1 = 1$, $a_2 = 2$, and $a_3 = 4$.

We now find a_n for general n. Since $-1 \leq f(n) - n \leq 2$, there are only two possible values for $f(n)$.

- Case 1: $f(n) = n$.

  There are clearly a_{n-1} ways to determine the other $n - 1$ values.

- Case 2: $f(n) = n - 1$.

  There are two possible values $1 \leq i \leq n - 1$ such that $f(i) = n$. The first value is $i = n - 1$. In that case, note that n and $n - 1$ are swapped by the bijection, so there are a_{n-2} ways to determine the other $n - 2$ values. The second value is $i = n - 2$. In that case, from the bound $-1 \leq f(n-1) - (n-1) \leq 2$, we know that $f(n-1)$ must be $n - 2$. Hence, our bijection cycles between n, $n - 1$, and $n - 2$, so there are a_{n-3} ways to determine the other $n - 3$ values.

Putting these two cases together, we have

$$a_n = a_{n-1} + a_{n-2} + a_{n-3},$$

for all $n \geq 4$.

Finally, we compute $a_4 = 7$, $a_5 = 13$, $a_6 = 24$, $a_7 = 44$, and $a_8 = 81$, as desired.