MOAA 2021: Gunga Bowl

October 16th, 2020

Gunga Bowl Problems

Gunga Bowl Set 1

G1. Evaluate $2 \times 0 + 2 \times 1 + 2 + 0 \times 2 + 1$.

G2. Add one pair of brackets to the expression

$$1 + 2 \times 3 + 4 \times 5 + 6$$

so that the resulting expression has a valid mathematical value, e.g., $1+2\times(3+4\times5)+6 = 53$. What is the largest possible value that one can make?

G3. What is the last digit of 2021^{2021}?

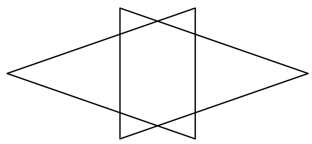
Gunga Bowl Set 2

G4. How many of the following capital English letters look the same when rotated 180° about their center?

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

G5. Joshua rolls two dice and records the product of the numbers face up. The probability that this product is composite can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

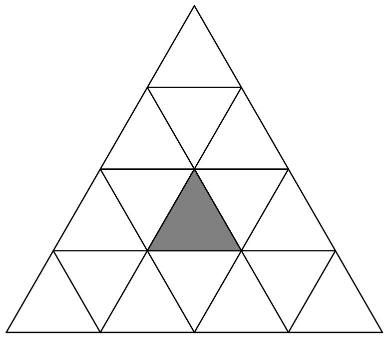
G6. Determine the number of triangles, of any size and shape, in the following figure:



Gunga Bowl Set 3

G7. Andover has a special weather forecast this week. On Monday, there is a $\frac{1}{2}$ chance of rain. On Tuesday, there is a $\frac{1}{3}$ chance of rain. This pattern continues all the way to Sunday, when there is a $\frac{1}{8}$ chance of rain. The probability that it doesn't rain in Andover all week can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

G8. Compute the number of triangles of different sizes which contain the gray triangle in the figure below.

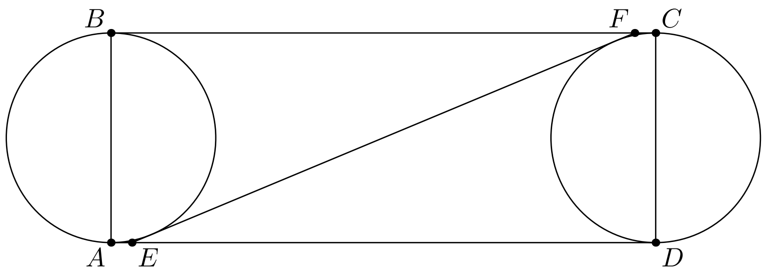


G9. William is biking from his home to his school and back, using the same route. When he travels to school, there is an initial 20° incline for 0.5 kilometers, a flat area for 2 kilometers, and a 20° decline for 1 kilometer. If William travels at 8 kilometers per hour during uphill 20° sections, 16 kilometers per hours during flat sections, and 20 kilometers per hour during downhill 20° sections, find the closest integer to the number of minutes it take William to get to school and back.

Gunga Bowl Set 4

G10. We say that an ordered pair (a, b) of positive integers with $a > b$ is *square-ish* if both $a+b$ and $a-b$ are perfect squares. For example, $(17, 8)$ is square-ish because $17+8 = 25$ and $17-8 = 9$ are both perfect squares. How many square-ish pairs (a, b) with $a+b < 100$ are there?

G11. Let $ABCD$ be a rectangle with $AB = 10$ and $BC = 26$. Let ω_1 be the circle with diameter $\overline{AB}$ and ω_2 be the circle with diameter $\overline{CD}$. Suppose ℓ is a common internal tangent to ω_1 and ω_2 and that ℓ intersects AD and BC at E and F respectively. What is EF?



G12. Andy wishes to open an electronic lock with a keypad containing all digits from 0 to 9. He knows that the password registered in the system is 2469. Unfortunately, he is also aware that exactly two different buttons (but he does not know which ones) $\underline{a}$ and $\underline{b}$ on the keypad are broken − when $\underline{a}$ is pressed the digit b is registered in the system, and when $\underline{b}$ is pressed the digit a is registered in the system. Find the least number of attempts Andy needs to surely be able to open the lock.

Gunga Bowl Set 5

G13. Determine the greatest power of 2 that is a factor of $3^{15} + 3^{11} + 3^6 + 1$.

G14. Sinclair starts with the number 1. Every minute, he either squares his number or adds 1 to his number, both with equal probability. What is the expected number of minutes until his number is divisible by 3?

G15. Let a, b, c, d be the four roots of the polynomial

$$x^4 + 3x^3 - x^2 + x - 2.$$

Given that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{1}{2}$ and $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}=-\frac{3}{4}$, the value of

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3}$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m+n$.

Gunga Bowl Set 6

G16. Let $1, 7, 19, \ldots$ be the sequence of numbers such that for all integers $n \geq 1$, the average of the first n terms is equal to the nth perfect square. Compute the last three digits of the 2021st term in the sequence.

G17. Isosceles trapezoid $ABCD$ has side lengths $AB = 6$ and $CD = 12$, while $AD = BC$. It is given that O, the circumcenter of $ABCD$, lies in the interior of the trapezoid. The extensions of lines AD and BC intersect at T. Given that $OT = 18$, the area of $ABCD$ can be expressed as $a + b\sqrt{c}$ where a, b, and c are positive integers where c is not divisible by the square of any prime. Compute $a + b + c$.

G18. Find the largest positive integer n such that the number $(2n)!$ ends with 10 more zeroes than the number $n!$.

Note: We define $n! = n \times (n-1) \times \cdots \times 1$ for all positive integers n.

Gunga Bowl Set 7

G19. Let S be the set of triples (a, b, c) of non-negative integers with $a + b + c$ even. The value of the sum

$$\sum_{(a,b,c)\in S} \frac{1}{2^a 3^b 5^c}$$

can be expressed as $\frac{m}{n}$ for relative prime positive integers m and n. Compute $m + n$.

G20. In the interior of square $ABCD$ with side length 1, a point P is chosen such that the lines ℓ_1, ℓ_2 through P parallel to AC and BD, respectively, divide the square into four distinct regions, the smallest of which has area $\mathcal{R}$. The area of the region of all points P for which $\mathcal{R} \geq \frac{1}{6}$ can be expressed as $\frac{a-b\sqrt{c}}{d}$ where $\gcd(a, b, d) = 1$ and c is not divisible by the square of any prime. Compute $a + b + c + d$.

G21. King William is located at $(1, 1)$ on the coordinate plane. Every day, he chooses one of the eight lattice points closest to him and moves to one of them with equal probability. When he exits the region bounded by the x, y axes and $x + y = 4$, he stops moving and remains there forever. Given that after an arbitrarily large amount of time he must exit the region, the probability he ends up on $x + y = 4$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Gunga Bowl Set 8

G22. Let p and q be positive integers such that p is a prime, p divides $q - 1$, and $p + q$ divides $p^2 + 2020q^2$. Find the sum of the possible values of p.

G23. Let P be a point chosen on the interior of side $\overline{BC}$ of triangle $\triangle ABC$ with side lengths $\overline{AB} = 10, \overline{BC} = 10, \overline{AC} = 12$. If X and Y are the feet of the perpendiculars from P to the sides AB and AC, then the minimum possible value of $PX^2 + PY^2$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

G24. Freddy the Frog is situated at 1 on an infinitely long number line. On day n, where $n \geq 1$, Freddy can choose to hop 1 step to the right, stay where he is, or hop k steps to the left, where k is an integer at most $n+1$. After day 5, how many sequences of moves are there such that Freddy has landed on at least one negative number?

Gunga Bowl Set 9

This set consists of three estimation problems, with scoring schemes described.

G25. Estimate N, the number of emails received by `director@andovermathopen.com` between MOAA 2020 and MOAA 2021.

An estimate of e gets $\left\lceil \frac{90}{|N-e|+3} \right\rceil$ points.

G26. Let A be the number of MOAA competitors with the letter "e" in their name. Let B be the number of MOAA competitors with the letter "t" in their name. Estimate N, the product AB.

An estimate of e gets $\max\{0, 30 - \lfloor 6\log(|N-e|+1) \rfloor\}$ points.

G27. Estimate N, the largest prime which divides the sum of all MOAA team IDs.

An estimate of e gets $\max\{0, \lfloor 130 \cdot \min\left\{\frac{e}{N}, \frac{N}{e}\right\} \rfloor - 100\}$ points.