MOAA 2021: Team Round

October 16th, 2021

Rules

- No mathematical texts, notes, or online resources of any kind are permitted. Rely on your brain and those of your teammates!
- Compasses, protractors, rulers, straightedges, graph paper, blank scratch paper, and writing implements are generally permitted, so long as they are not designed to give an unfair advantage.
- No computational aids (including but not limited to calculators, phones, calculator watches, and computer programs) are permitted on any portion of the MOAA.
- Individual may only receive help from members of their team. Consulting any other individual is grounds for disqualification.

How to Compete

- After completing the test, your team captain should input your answers, along with your Team ID, into the provided Team Round Google Form.

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Team Round Problems

The Team Round consists of 20 problems, ordered in approximately increasing difficulty, to be solved in 45 minutes. All answers are nonnegative integers no larger than 1,000,000.

T1. The value of

$$\frac{1}{20} - \frac{1}{21} + \frac{1}{20 \times 21}$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

T2. Four students Alice, Bob, Charlie, and Diana want to arrange themselves in a line such that Alice is at either end of the line, i.e., she is not in between two students. In how many ways can the students do this?

T3. For two real numbers x and y, let $x \circ y = \frac{xy}{x+y}$. The value of

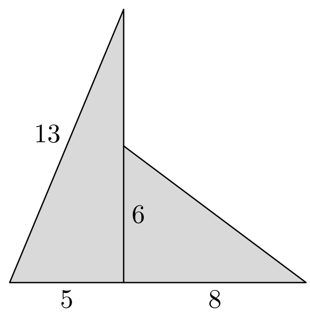
$$1 \circ (2 \circ (3 \circ (4 \circ 5)))$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

T4. Compute the number of ordered triples (x, y, z) of integers satisfying

$$x^2 + y^2 + z^2 = 9.$$

T5. Two right triangles are placed next to each other to form a quadrilateral as shown. What is the perimeter of the quadrilateral?



T6. Find the sum of all two-digit prime numbers whose digits are also both prime numbers.

T7. Compute the number of ordered pairs (a, b) of positive integers satisfying $a^b = 2^{100}$.

T8. Evaluate

$$2^7 \times 3^0 + 2^6 \times 3^1 + 2^5 \times 3^2 + \cdots + 2^0 \times 3^7.$$

T9. Mr. DoBa has a bag of markers. There are 2 blue, 3 red, 4 green, and 5 yellow markers. Mr. DoBa randomly takes out two markers from the bag. The probability that these two markers are different colors can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

T10. For how many nonempty subsets $S \subseteq \{1, 2, \ldots, 10\}$ is the sum of all elements in S even?

T11. Find the product of all possible real values for k such that the system of equations

$$\begin{cases} x^2 + y^2 = 80 \\ x^2 + y^2 = k + 2x - 8y \end{cases}$$

has exactly one real solution (x, y).

T12. Let $\triangle ABC$ have $AB = 9$ and $AC = 10$. A semicircle is inscribed in $\triangle ABC$ with its center on segment BC such that it is tangent AB at point D and AC at point E. If $AD = 2DB$ and r is the radius of the semicircle, r^2 can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

T13. Bob has 30 identical unit cubes. He can join two cubes together by gluing a face on one cube to a face on the other cube. He must join all the cubes together into one connected solid. Over all possible solids that Bob can build, what is the largest possible surface area of the solid?

T14. Evaluate

$$\left\lfloor \frac{1 \times 5}{7} \right\rfloor + \left\lfloor \frac{2 \times 5}{7} \right\rfloor + \left\lfloor \frac{3 \times 5}{7} \right\rfloor + \cdots + \left\lfloor \frac{100 \times 5}{7} \right\rfloor.$$

Note: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

T15. Consider the polynomial

$$P(x) = x^3 + 3x^2 + 6x + 10.$$

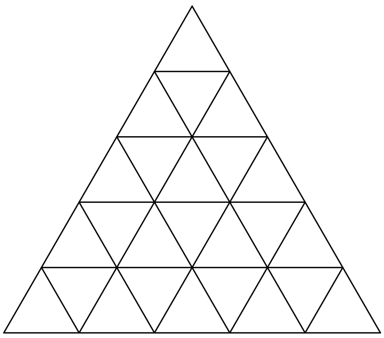
Let its three roots be a, b, c. Define $Q(x)$ to be the monic cubic polynomial with roots ab, bc, ca. Compute $|Q(1)|$.

T16. Let $\triangle ABC$ have $\angle ABC = 67^\circ$. Point X is chosen such that $AB = XC$, $\angle XAC = 32^\circ$, and $\angle XCA = 35^\circ$. Compute $\angle BAC$ in degrees.

T17. Compute the remainder when 10^{2021} is divided by 10101.

T18. Let $\triangle ABC$ be a triangle with side length $BC = 4\sqrt{6}$. Denote ω as the circumcircle of $\triangle ABC$. Point D lies on ω such that AD is the diameter of ω. Let N be the midpoint of arc BC that contains A. H is the intersection of the altitudes in $\triangle ABC$ and it is given that $HN = HD = 6$. If the area of $\triangle ABC$ can be expressed as $\frac{a\sqrt{b}}{c}$, where a, b, c are positive integers with a and c relatively prime and b not divisible by the square of any prime, compute $a + b + c$.

T19. Consider the 5 by 5 by 5 equilateral triangular grid as shown:



Ethan chooses two distinct upward-oriented equilateral triangles bounded by the gridlines. The probability that Ethan chooses two triangles that share exactly one vertex can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

T20. Compute the sum of all integers x for which there exists an integer y such that

$$x^3 + xy + y^3 = 503.$$