MOAA 2021 Accuracy Round Solutions

Math Open at Andover

October 16, 2021

A1. Evaluate

$$2 \times (2 \times (2 \times (2 \times (2 \times (2 \times 2 - 2) - 2) - 2) - 2) - 2) - 2.$$

Proposed by: Nathan Xiong

Answer: $\boxed{2}$

Solution: Note that no matter how many times we unravel from the inside, the expression is always 2 since $2 \times 2 - 2 = 2$. Hence our final answer is 2.

A2. On Andover's campus, Graves Hall is 60 meters west of George Washington Hall, and George Washington Hall is 80 meters north of Paresky Commons. Jessica wants to walk from Graves Hall to Paresky Commons. If she first walks straight from Graves Hall to George Washington Hall and then walks straight from George Washington Hall to Paresky Commons, it takes her 8 minutes and 45 seconds while walking at a constant speed. If she walks with the same speed directly from Graves Hall to Paresky Commons, how much time does she save, in seconds?

Proposed by: Nathan Xiong

Answer: $\boxed{150}$

Solution: It takes Jessica 525 seconds to walk a total of $60 + 80 = 140$ meters. Therefore, her speed is $\frac{140}{525}$ meters per second. By the Pythagorean Theorem, the direct distance is $\sqrt{60^2 + 80^2} = 100$, so if she walks at the same speed, then

$$\frac{140 \text{ meters}}{525 \text{ seconds}} = \frac{100 \text{ meters}}{t \text{ seconds}},$$

where t is time in seconds it takes to travel directly. Solving yields $t = 375$, so she saves $525 - 375 = 150$ seconds.

A3. Arnav is placing three rectangles into a 3×3 grid of unit squares. He has a 1×3 rectangle, a 1×2 rectangle, and a 1×1 rectangle. He must place the rectangles onto the grid such that the edges of the rectangles align with the gridlines of the grid. If he is allowed to rotate the rectangles, how many ways can he place the three rectangles into the grid, without overlap?

Proposed by: William Yue

Answer: $\boxed{144}$

Solution: We do cases on where Arnav puts the 1×3 rectangle.

There are 4 places to put the 1×3 rectangle against a boundary of the grid, and these 4 cases are symmetric. After placing the first rectangle, notice that there are 7 ways to ways to place the 2×1 rectangle and then 4 ways to place the 1×1 rectangle. Thus, this contributes $4 \times 7 \times 4 = 112$ ways.

There are 2 places to put the 1×3 rectangle across the middle of the grid, and these 2 cases are symmetric. After placing the first rectangle, notice that there are 4 ways to place the 2×1 rectangle and then 4 ways to place the 1×1 rectangle. Thus, this contributes $2 \times 4 \times 4 = 32$ ways.

Summing yields a total of $112 + 32 = 144$ ways.

A4. Compute the number of two-digit numbers $\overline{ab}$ with nonzero digits a and b such that a and b are both factors of $\overline{ab}$.

Proposed by: Nathan Xiong

Answer: $\boxed{14}$

Solution: The number $\overline{ab}$ has value $10a + b$, so we are given that $a \mid 10a + b \implies a \mid b$ and $b \mid 10a + b \implies b \mid 10a$. So we know $b = ka$ for some integer k where $k \mid 10$. Clearly, $k = 10$ is impossible, so there are three possible cases:

- If $k = 1$, then $11, 22, \ldots, 99$ all work, giving 9 numbers.
- If $k = 2$, then $12, 24, 36, 48$ all work, giving 4 numbers.
- If $k = 5$, then 15 works, giving 1 number.

Summing, the answer is $9 + 4 + 1 = 14$.

A5. If x, y, z are nonnegative integers satisfying the equation below, then compute $x + y + z$.

$$\left(\frac{16}{3}\right)^x \times \left(\frac{27}{25}\right)^y \times \left(\frac{5}{4}\right)^z = 256.$$

Proposed by: Jeffrey Shi

Answer: $\boxed{6}$

Solution: Expand everything into primes

$$(2^4 \times 3^{-1})^x \times (3^3 \times 5^{-2})^y \times (5 \times 2^{-2})^z = 2^8 \times 3^0 \times 5^0.$$

This can be rewritten as

$$2^{4x-2z} \times 3^{-x+3y} \times 5^{-2y+z} = 2^8 \times 3^0 \times 5^0.$$

Equating all exponents results in a system of equations $4x - 2z = 8, x = 3y, z = 2y$, and solving yields $x = 3, y = 1, z = 2$. The answer is $3 + 1 + 2 = 6$.

A6. Let $\triangle ABC$ be a triangle in a plane such that $AB = 13$, $BC = 14$, and $CA = 15$. Let D be a point in three-dimensional space such that $\angle BDC = \angle CDA = \angle ADB = 90^\circ$. Let d be the distance from D to the plane containing $\triangle ABC$. The value d^2 can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by: William Yue

Answer: $\boxed{511}$

Solution: We will first compute the volume of tetrahedron $ABCD$. Since we can determine the area of $\triangle ABC$, this will allow us to find the length of the height of the tetrahedron from D to $\triangle ABC$.

Let $AD = x$, $BD = y$, and $CD = z$. By the Pythagorean Theorem on triangle $\triangle ABD$, we get that

$$AD^2 + BD^2 = AB^2 \implies x^2 + y^2 = 13^2.$$

Similarly, we get $y^2 + z^2 = 14^2$ and $z^2 + x^2 = 15^2$. Adding these three equations together and dividing by two gives

$$x^2 + y^2 + z^2 = \frac{13^2 + 14^2 + 15^2}{2} = 295.$$

Now, subtracting the previous three equations from this gives $x^2 = 295 - 14^2 = 99$, $y^2 = 295 - 15^2 = 70$, and $z^2 = 295 - 13^2 = 126$. Therefore, the volume of tetrahedron $ABCD$ is

$$\frac{xyz}{6} = \frac{\sqrt{99 \cdot 70 \cdot 126}}{6} = \sqrt{11 \cdot 35 \cdot 63} = 21\sqrt{55}.$$

We can compute the area of $\triangle ABC$ using Heron's formula as $\sqrt{21 \cdot 6 \cdot 7 \cdot 8} = 84$. Therefore,

$$\frac{84d}{3} = 21\sqrt{55} \implies d = \frac{21\sqrt{55}}{28} = \frac{3\sqrt{55}}{4}.$$

Thus, $d^2 = \frac{9 \cdot 55}{16} = \frac{495}{16}$, so the answer is $495 + 16 = 511$.

A7. Jeffrey rolls fair three six-sided dice and records their results. The probability that the mean of these three numbers is greater than the median of these three numbers can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by: Nathan Xiong

Answer: $\boxed{101}$

Solution: By symmetry, the probability that the mean is greater than the median is equal to the probability that the mean is less than the median. So it suffices to find the probability that the mean is equal to the median, which happens if and only if the three rolls form an arithmetic sequence. We split cases:

- The arithmetic sequence is constant. Clearly, there are 6 ways this can happen.
- The arithmetic sequence has common difference 1. Then there are 4 possible sequences, with 6 permutations for each, so 24 ways.
- The arithmetic sequence has common difference 2. Then there are 2 possible sequences, with 6 permutations for each, so 12 ways.

Summing, there are 42 ways to roll the three dice so that the mean is equal to the median. In total, there are $6^3 = 216$ ways to roll the three dice. So, the probability that the mean is greater than the median is $\frac{216-42}{2\times 216} = \frac{29}{72}$. The answer is $29 + 72 = 101$.

A8. Will has a magic coin that can remember previous flips. If the coin has already turned up heads m times and tails n times, the probability that the next flip turns up heads is exactly $\frac{m+1}{m+n+2}$. Suppose that the coin starts at 0 flips. The probability that after 10 coin flips, heads and tails have both turned up exactly 5 times can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by: Nathan Xiong

Answer: $\boxed{12}$

Solution: Note that no matter what the final outcome of heads and tails is, the probability of it happening is always $\frac{5!\cdot 5!}{11!}$, due to the nature of the probability function $\frac{m+1}{m+n+2}$. There are $\binom{10}{5}$ ways to flip 5 heads and 5 tails, so our final probability is

$$\binom{10}{5}\cdot\frac{5!\cdot 5!}{11!}=\frac{1}{11}.$$

The answer is $1+11=12$.

A9. Let S be the set of ordered pairs (x, y) of positive integers for which $x+y\le 20$. Evaluate

$$\sum_{(x,y)\in S}(-1)^{x+y}xy.$$

Proposed by: Andrew Wen

Answer: $\boxed{715}$

Solution: Let $x+y=d$ and rewrite the sum as

$$\sum_{d=2}^{20}(-1)^d\left(\sum_{x,y\ge 1,x+y=d}xy\right).$$

It is easy to check that for all d,

$$\sum_{x,y\ge 1,x+y=d}xy=\binom{d+1}{3}.$$

The sum then simplifies into

$$\sum_{d=2}^{20}(-1)^d\binom{d+1}{3}=\binom{3}{3}-\binom{4}{3}+\binom{5}{3}-\cdots+\binom{21}{3}.$$

We can use the Hockey Stick identity to simplify the expression into

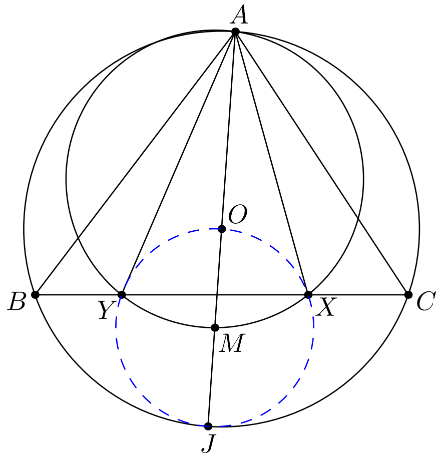
$$1+\sum_{i=1}^{9}\left[\binom{2i+3}{3}-\binom{2i+2}{3}\right]=\sum_{i=1}^{10}\binom{2i}{2}=715.$$

A10. In $\triangle ABC$, let X and Y be points on segment BC such that $AX=XB=20$ and $AY=YC=21$. Let J be the A-excenter of triangle $\triangle AXY$. Given that J lies on the circumcircle of $\triangle ABC$, the length of BC can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m+n$.

Proposed by: Andrew Wen

Answer: $\boxed{85}$

Solution: Focus on $\triangle AXY$. Note that the angle bisectors of X and Y are in fact the perpendicular bisectors of AB and AC. Hence, the incenter of $\triangle AXY$ is just O, the circumcenter of $\triangle ABC$.



We know that J lies on AO, the $\angle XAY$ angle bisector. Since J lies on (ABC), it follows that J must be the A-antipode, and thus $JO = AO$.

Now, let M be the midpoint of OJ, which by the Incenter-Excenter lemma is the midpoint of minor arc $\widehat{XY}$ in (AXY). If we let $\ell = MO = MJ$, note that $AO = OJ = 2\ell$ and by Ptolemy on $AXMY$, we have

$$\ell \cdot (AX + AY) = 3\ell \cdot XY.$$

Hence, $XY = \frac{1}{3}(AX + AY) = \frac{41}{3}$. We can finally extract the answer by noting that

$$BC = BX + CY - XY = AX + AY - XY = 20 + 21 - \frac{41}{3} = \frac{82}{3}.$$

The answer is $82 + 3 = 85$.