MOAA 2021: Gunga Bowl Solutions

October 16th, 2021

Gunga Bowl Set 1

G1. Evaluate $2 \times 0 + 2 \times 1 + 2 + 0 \times 2 + 1$.

Proposed by: Nathan Xiong

Answer: $\boxed{5}$

Solution: This is just

$$0 + 2 + 2 + 0 + 1 = 5.$$

G2. Add one pair of brackets to the expression

$$1 + 2 \times 3 + 4 \times 5 + 6$$

so that the resulting expression has a valid mathematical value, e.g., $1+2\times(3+4\times5)+6 = 53$. What is the largest possible value that one can make?

Proposed by: Nathan Xiong

Answer: $\boxed{77}$

Solution: Intuitively, we want more multiplications together. So,

$$1 + 2 \times (3 + 4) \times 5 + 6 = 1 + 2 \times 7 \times 5 + 6 = 77$$

yields the maximum value.

G3. What is the last digit of 2021^{2021}?

Proposed by: Yifan Kang

Answer: $\boxed{1}$

Solution: Note that $2021^{2021} \equiv 1^{2021} \equiv 1 \pmod{10}$, so our answer is 1.

Gunga Bowl Set 2

G4. How many of the following capital English letters look the same when rotated 180° about their center?

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Proposed by: William Yue

Answer: $\boxed{7}$

Solution: A quick check shows that the desired letters are {H, I, N, O, S, X, Z}, of which there are 7.

G5. Joshua rolls two dice and records the product of the numbers face up. The probability that this product is composite can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by: Nathan Xiong

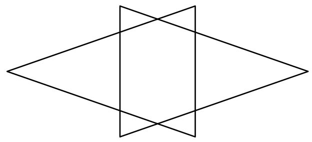
Answer: $\boxed{65}$

Solution: Consider the complement. The product is prime if and only if one roll is 1 and the other roll is prime, which happens with probability $\frac{3+3}{36} = \frac{1}{6}$. We also have to watch out when we roll two 1s, since 1 is neither prime nor composite. This happens with probability $\frac{1}{36}$, so our final probability is

$$1 - \frac{1}{6} - \frac{1}{36} = \frac{29}{36}.$$

The answer is $29 + 36 = 65$.

G6. Determine the number of triangles, of any size and shape, in the following figure:



Proposed by: William Yue

Answer: $\boxed{8}$

Solution: By carefully counting, we see that there are 4 small triangles, 2 medium triangles, and 2 large triangles, for a total of $4 + 2 + 2 = 8$ triangles.

Gunga Bowl Set 3

G7. Andover has a special weather forecast this week. On Monday, there is a $\frac{1}{2}$ chance of rain. On Tuesday, there is a $\frac{1}{3}$ chance of rain. This pattern continues all the way to Sunday, when there is a $\frac{1}{8}$ chance of rain. The probability that it doesn't rain in Andover all week can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by: Nathan Xiong

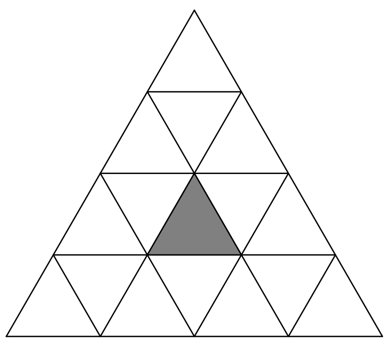
Answer: $\boxed{9}$

Solution: The desired probability is just

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{8}\right) = \frac{1}{2} \times \frac{2}{3} \times \cdots \times \frac{7}{8} = \frac{1}{8}.$$

The answer is $1 + 8 = 9$.

G8. Compute the number of triangles of different sizes which contain the gray triangle in the figure below.



Proposed by: Nathan Xiong

Answer: $\boxed{9}$

Solution: We do cases on the size of the triangle containing the center:

- There is 1 triangle with side length 1, namely the gray triangle itself.
- There are 4 triangles with side length 2, including a downward pointing one.
- There are 3 triangles with side length 3.
- There is 1 triangle with side length 4, namely the big triangle itself.

The answer is $1 + 4 + 3 + 1 = 9$.

G9. William is biking from his home to his school and back, using the same route. When he travels to school, there is an initial 20° incline for 0.5 kilometers, a flat area for 2 kilometers, and a 20° decline for 1 kilometer. If William travels at 8 kilometers per hour during uphill 20° sections, 16 kilometers per hours during flat sections, and 20 kilometers per hour during downhill 20° sections, find the closest integer to the number of minutes it take William to get to school and back.

Proposed by: William Yue

Answer: $\boxed{31}$

Solution: Going to school and back, William travels a total distance of 1.5km uphill, 1.5km downhill, and 4km flat (note that an uphill path to school becomes a downhill path back home and vice versa). Since time $= \frac{\text{distance}}{\text{speed}}$, the total time taken in hours is

$$\frac{1.5}{20} + \frac{1.5}{8} + \frac{4}{16} = \frac{41}{80} \text{ hours,}$$

which becomes $\frac{123}{4} = 30.75$ minutes. This rounds to an answer of 31.

Gunga Bowl Set 4

G10. We say that an ordered pair (a, b) of positive integers with $a > b$ is *square-ish* if both $a+b$ and $a-b$ are perfect squares. For example, $(17, 8)$ is square-ish because $17 + 8 = 25$ and $17 - 8 = 9$ are both perfect squares. How many square-ish pairs (a, b) with $a + b < 100$ are there?

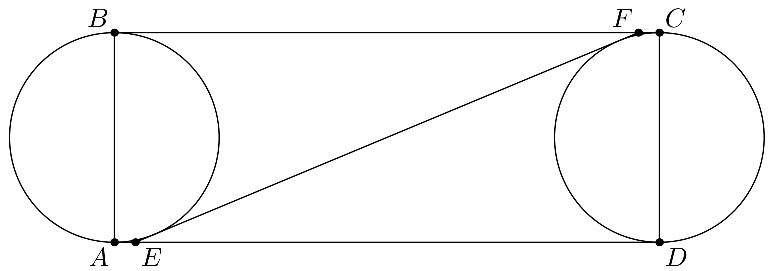
Proposed by: Nathan Xiong

Answer: $\boxed{16}$

Solution: For some integers x, y, we have $a + b = x^2$ and $a - b = y^2$. So, $(a, b) = \left(\frac{x^2+y^2}{2}, \frac{x^2-y^2}{2}\right)$. Note that x^2 is a perfect square less than 100, and y^2 is a perfect square less than x^2 but of the same parity. Checking the nine possible values $x^2 = 1, 4, 9, \ldots, 81$, we get that the answer is

$$4 + 3 + 3 + 2 + 2 + 1 + 1 = 16.$$

G11. Let $ABCD$ be a rectangle with $AB = 10$ and $BC = 26$. Let ω_1 be the circle with diameter $\overline{AB}$ and ω_2 be the circle with diameter $\overline{CD}$. Suppose ℓ is a common internal tangent to ω_1 and ω_2 and that ℓ intersects AD and BC at E and F respectively. What is EF?



Proposed by: Nathan Xiong

Answer: $\boxed{26}$

Solution: Let EF be tangent to ω_1 at T. Then by congruent triangles,

$$EF = ET + FT = ET + FB = EA + FB = FC + FB = BC = 26.$$

G12. Andy wishes to open an electronic lock with a keypad containing all digits from 0 to 9. He knows that the password registered in the system is 2469. Unfortunately, he is also aware that exactly two different buttons (but he does not know which ones) $\underline{a}$ and $\underline{b}$ on the keypad are broken − when $\underline{a}$ is pressed the digit b is registered in the system, and when $\underline{b}$ is pressed the digit a is registered in the system. Find the least number of attempts Andy needs to surely be able to open the lock.

Proposed by: Andrew Wen

Answer: $\boxed{31}$

Solution: If $a, b \notin \{2, 4, 6, 9\}$, then he just needs to try 2469. If $a, b \in \{2, 4, 6, 9\}$, then he just needs to try the 6 passwords formed by swapping two buttons in 2469. If one of a, b is in $\{2, 4, 6, 9\}$ and the other is not, then he needs to try $4 \cdot 6 = 24$ passwords, where each time he tries replacing one of $\{2, 4, 6, 9\}$ with one of $\{0, 1, 3, 5, 7, 8\}$, covering all 24 possible replacements in 24 tries.

Therefore, it takes $1 + 6 + 24 = 31$ tries to cover all possible keypad sequences that may possibly register the correct password.

Gunga Bowl Set 5

G13. Determine the greatest power of 2 that is a factor of $3^{15} + 3^{11} + 3^6 + 1$.

Proposed by: Nathan Xiong

Answer: $\boxed{64}$

Solution: Note that

$$3^{15} + 3^{11} + 3^6 + 1 = (3^5 + 1)^3 = 244^3 = 2^6 \times 61^3.$$

So, the answer is just $2^6 = 64$.

G14. Sinclair starts with the number 1. Every minute, he either squares his number or adds 1 to his number, both with equal probability. What is the expected number of minutes until his number is divisible by 3?

Proposed by: Nathan Xiong

Answer: $\boxed{6}$

Solution: Let E_i be the expected number of rolls needed to reach a multiple of 3 starting from i mod 3. We get the "states" equations

$$E_1 = \frac{1}{2}(E_1 + 1) + \frac{1}{2}(E_2 + 1),$$
$$E_2 = \frac{1}{2} + \frac{1}{2}(E_1 + 1).$$

Solving this system of equations gives $E_1 = 6$.

G15. Let a, b, c, d be the four roots of the polynomial

$$x^4 + 3x^3 - x^2 + x - 2.$$

Given that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{2}$ and $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} = -\frac{3}{4}$, the value of

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3}$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by: Nathan Xiong

Answer: $\boxed{39}$

Solution: Note that for every root r,

$$r^4 + 3r^3 - r^2 + r = 2 \implies r + 3 - \frac{1}{r} + \frac{1}{r^2} = \frac{2}{r^3}.$$

By Vieta's formulas, the sum of the roots is $a+b+c+d = -3$. Then, cyclically summing over the four roots,

$$\sum_{\text{cyc}} \frac{2}{a^3} = \sum_{\text{cyc}} \left(a + 3 - \frac{1}{a} + \frac{1}{a^2}\right) = -3 + 12 - \frac{1}{2} - \frac{3}{4} = \frac{31}{4}.$$

The desired sum is then $\frac{31}{8}$, and the answer is $31 + 8 = 39$.

Gunga Bowl Set 6

G16. Let $1, 7, 19, \ldots$ be the sequence of numbers such that for all integers $n \geq 1$, the average of the first n terms is equal to the nth perfect square. Compute the last three digits of the 2021st term in the sequence.

Proposed by: Nathan Xiong

Answer: $\boxed{261}$

Solution: Since the average of the first n terms is n^2, their sum is n^3. If we denote our sequence by $a_1, a_2, a_3, \ldots$, and we define the sequence $S_1, S_2, S_3, \ldots$ by taking the partial sums, it follows that

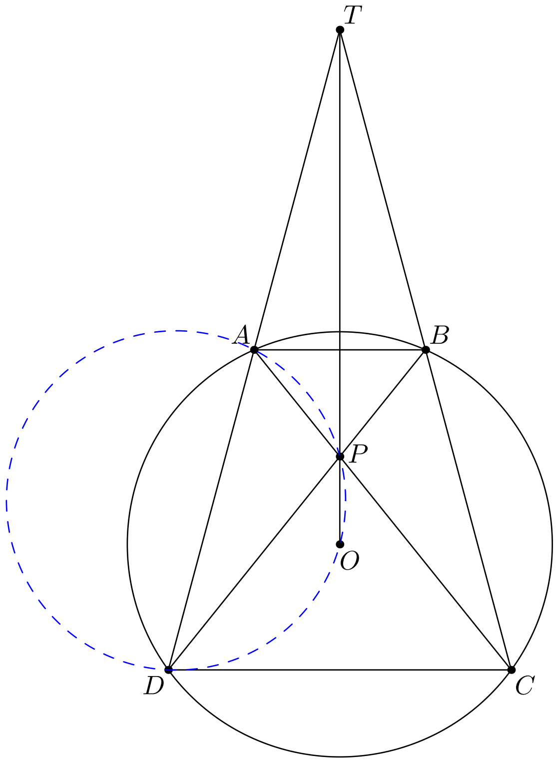
$$a_{2021} = S_{2021} - S_{2020} = 2021^3 - 2020^3 \equiv 21^3 \equiv 261 \pmod{1000}.$$

G17. Isosceles trapezoid $ABCD$ has side lengths $AB = 6$ and $CD = 12$, while $AD = BC$. It is given that O, the circumcenter of $ABCD$, lies in the interior of the trapezoid. The extensions of lines AD and BC intersect at T. Given that $OT = 18$, the area of $ABCD$ can be expressed as $a + b\sqrt{c}$ where a, b, and c are positive integers where c is not divisible by the square of any prime. Compute $a + b + c$.

Proposed by: Andrew Wen

Answer: $\boxed{84}$

Solution: Let $AC \cap BD = P$. Note that $\angle ADP = \frac{1}{2}\angle AOB = \angle AOP$, so $APOD$ is cyclic.



Let h be the height of the trapezoid. We know by similar triangle ratios that the height from T to AB is h and that the height from P to AB is $\frac{h}{3}$. By Power of a Point from T to $(ADOP)$, we have

$$TA \cdot TD = TP \cdot TO = \sqrt{(h^2+9)(4h^2+36)} = \frac{4h}{3} \cdot 18.$$

This simplifies to

$$2(h^2+9) = 24h \implies h^2 - 12h + 9 = 0 \implies h = 6 \pm 3\sqrt{3}.$$

We take the larger solution, since if h is too small, then O lies outside $ABCD$. The area of the trapezoid is

$$\frac{1}{2}(6+12)h = 9h = 54 + 27\sqrt{3}.$$

The answer is $54 + 27 + 3 = 84$.

G18. Find the largest positive integer n such that the number $(2n)!$ ends with 10 more zeroes than the number $n!$.

Note: We define $n! = n \times (n-1) \times \cdots \times 1$ for all positive integers n.

Proposed by: Andy Xu

Answer: $\boxed{42}$

Solution: We need to make the number of powers of 5 dividing $(n+1)(n+2)\cdots(2n)$ at least 10. A rough estimate is that there is a power of 5 every 5 numbers, so $n \approx 50$. But this is an overestimate because it is possible for a number to provide multiple factors of 5. Note that

$$45 \times 50 \times \ldots \times 80$$

provides us with the necessary amount of factors of 5, and if we went any higher (i.e included 85), we would have extra factors of 5. So, the largest n that works is just $\lfloor \frac{85}{2} \rfloor = 42$.

Gunga Bowl Set 7

G19. Let S be the set of triples (a, b, c) of non-negative integers with $a + b + c$ even. The value of the sum

$$\sum_{(a,b,c)\in S} \frac{1}{2^a 3^b 5^c}$$

can be expressed as $\frac{m}{n}$ for relative prime positive integers m and n. Compute $m + n$.

Proposed by: Nathan Xiong

Answer: $\boxed{37}$

Solution: There are two cases for (a, b, c):

- Case 1: All of a, b, c are even. Then, the sum becomes

$$\sum_{a,b,c} \frac{1}{4^a 9^b 25^c} = \left(1 + \frac{1}{4} + \dots\right)\left(1 + \frac{1}{9} + \dots\right)\left(1 + \frac{1}{25} + \dots\right) = \frac{4}{3} \times \frac{9}{8} \times \frac{25}{24} = \frac{25}{16}.$$

- Case 2: Two of a, b, c are odd. Then, the sum becomes

$$\left(\frac{1}{2 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 2}\right) \sum_{a,b,c} \frac{1}{4^a 9^b 25^c} = \frac{1}{3} \times \frac{25}{16}.$$

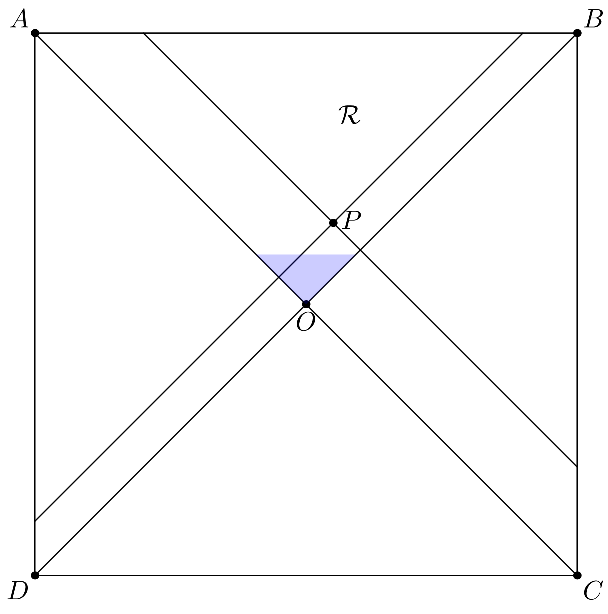
So, the final sum is $\frac{4}{3} \times \frac{25}{16} = \frac{25}{12}$, and the answer is $25 + 12 = 37$.

G20. In the interior of square $ABCD$ with side length 1, a point P is chosen such that the lines ℓ_1, ℓ_2 through P parallel to AC and BD, respectively, divide the square into four distinct regions, the smallest of which has area $\mathcal{R}$. The area of the region of all points P for which $\mathcal{R} \geq \frac{1}{6}$ can be expressed as $\frac{a-b\sqrt{c}}{d}$ where $\gcd(a, b, d) = 1$ and c is not divisible by the square of any prime. Compute $a + b + c + d$.

Proposed by: Andrew Wen

Answer: $\boxed{16}$

Solution: Let O be the center of $ABCD$. Assume without loss of generality that point P is in $\triangle OAB$. Suppose ℓ_1 and ℓ_2 intersect side AB at X and Y respectively. Let d be the distance from P to AB; by isosceles triangles, $XY = 2d$.



Now, note that XPY must be the smallest among the four relevant regions, and it has area $\frac{1}{2} \cdot 2d \cdot d = d^2$. So, we want to find the area of the region in $\triangle OAB$ with $d \geq \frac{1}{\sqrt{6}}$. It's easy to see that this is a triangle with height $h = \frac{1}{2} - \frac{1}{\sqrt{6}}$ and base $2h$.

By symmetry, the other three cases when P lies in $\triangle OBC$, $\triangle OCD$, and $\triangle ODA$ give the same area. So, the total area is

$$4h^2 = \left(1 - \frac{\sqrt{6}}{3}\right)^2 = \frac{5 - 2\sqrt{6}}{3},$$

as desired. The answer is $5 + 2 + 6 + 3 = 16$.

G21. King William is located at $(1, 1)$ on the coordinate plane. Every day, he chooses one of the eight lattice points closest to him and moves to one of them with equal probability. When he exits the region bounded by the x, y axes and $x + y = 4$, he stops moving and remains there forever. Given that after an arbitrarily large amount of time he must exit the region, the probability he ends up on $x + y = 4$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{65}$

Solution: There are three lattice points enclosed within the region; $(1, 1), (1, 2), (2, 1)$. Let P be the probability to end up on $x + y = 4$ from $(1, 1)$, and let Q be the probability to end up on $x + y = 4$ from $(1, 2)$ and $(2, 1)$ (these two probabilities are identical by symmetry). We may write

$$P = \frac{1}{4}Q + \frac{1}{8},$$
$$Q = \frac{1}{4} + \frac{1}{8}P + \frac{1}{8}Q.$$

Solving this system of equations yields $P = \frac{11}{54}$. The answer is $11 + 54 = 65$.

Gunga Bowl Set 8

G22. Let p and q be positive integers such that p is a prime, p divides $q - 1$, and $p + q$ divides $p^2 + 2020q^2$. Find the sum of the possible values of p.

Proposed by: Andy Xu

Answer: $\boxed{141}$

Solution: Note that $p+q \mid p^2+2020q^2$ implies $p+q \mid p^2+2020q^2-(p+q)(p-q) = 2021q^2$. Since $p \mid q - 1$ we know $\gcd(p, q) = 1$. Thus, $p + q \mid 2021$. Let $q = pn + 1$ for a positive integer n. We have

$$p(n + 1) + 1 \mid 2021.$$

Consequently, $p(n + 1)$ can be 42, 46, or 2020. When $p(n + 1) = 42$, we have $p = 2, 3, 7$. When $p(n + 1) = 46$, we have $p = 2, 23$. When $p(n + 1) = 2020$ we have $p = 2, 5, 101$. Combining all these solutions, $p = 2, 3, 5, 7, 23, 101$, which sum to 141. It's easy to check that for each of these p there exists a valid q.

G23. Let P be a point chosen on the interior of side $\overline{BC}$ of triangle $\triangle ABC$ with side lengths $\overline{AB} = 10, \overline{BC} = 10, \overline{AC} = 12$. If X and Y are the feet of the perpendiculars from P to the sides AB and AC, then the minimum possible value of $PX^2 + PY^2$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{2365}$

Solution: Note that

$$[ABC] = [PAB] + [PAC] = 5PX + 6PY.$$

By Cauchy-Schwarz, we have $(25+36)(PX^2+PY^2) \geq [ABC]^2$. Finally, note that $\triangle ABC$ is formed by two 6-8-10 triangles, so we easily get $[ABC] = 2 \cdot 24 = 48$. The value of $PX^2 + PY^2$ is then at least $\frac{48^2}{61} = \frac{2304}{61}$, and the answer is $2304 + 61 = 2365$. Note that equality occurs when $6PX = 5PY$.

G24. Freddy the Frog is situated at 1 on an infinitely long number line. On day n, where $n \geq 1$, Freddy can choose to hop 1 step to the right, stay where he is, or hop k steps to the left, where k is an integer at most $n+1$. After day 5, how many sequences of moves are there such that Freddy has landed on at least one negative number?

Proposed by: Andy Xu

Answer: $\boxed{6423}$

Solution: We instead count the number of ways Freddy can hop so that he is always on a nonnegative number. Let $O(n, m)$ be the number of ways Freddy can do this if he is on $m \geq 0$ on the number line after n moves. Since Freddy can only move right at most n times, for all $m > n+1$, we have $O(n, m) = 0$. So, we want to find $O(5, 0)+O(5, 1)+\cdots+O(5, 6)$.

We'll derive a recurrence relation for $O(n, m)$. Consider Freddy's move on day n. If he moves right, he was at $m-1$ on day $n-1$. If he stays still, he was at m on day $n-1$. If he moves left, he was on one of $m+1, m+2, \ldots, m+n+1$ on day $n-1$. So, we have the recurrence relation

$$O(n, m) = O(n-1, m-1) + O(n-1, m) + O(n-1, m+1) + \cdots + O(n-1, m+n+1).$$

And since $O(n, m) = 0$ for all $m > n+1$, we can simplify this to

$$O(n, m) = O(n-1, m-1) + O(n-1, m) + O(n-1, m+1) + \cdots + O(n-1, n).$$

Note that we always take $O(n, -1) = 0$. From here, the problem is just computation. It's easy to get from this recurrence relation that

$$O(5, 0) + O(5, 1) + \cdots + O(5, 6) = 297.$$

The total number of ways that Freddy can hop is $4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 = 6720$, so the answer is $6720 - 297 = 6423$.

Gunga Bowl Set 9

This set consists of three estimation problems, with scoring schemes described.

G25. Estimate N, the number of emails received by `director@andovermathopen.com` between MOAA 2020 and MOAA 2021.

An estimate of e gets $\left\lceil \frac{90}{|N-e|+3} \right\rceil$ points.

Proposed by: Nathan Xiong

Answer: $\boxed{268}$

G26. Let A be the number of MOAA competitors with the letter "e" in their name. Let B be the number of MOAA competitors with the letter "t" in their name. Estimate N, the product AB.

An estimate of e gets $\max\{0, 30 - \lfloor 6\log(|N-e|+1)\rfloor\}$ points.

Proposed by: William Yue

Answer: $\boxed{175725}$

G27. Estimate N, the largest prime which divides the sum of all MOAA team IDs.

An estimate of e gets $\max\{0, \lfloor 130 \cdot \min\left\{\frac{e}{N}, \frac{N}{e}\right\}\rfloor - 100\}$ points.

Proposed by: Nathan Xiong

Answer: $\boxed{4933}$