MOAA 2021 Speed Round Solutions

Math Open at Andover

October 16, 2021

S1. What is $2021 + 20 + 21 + 2 + 0 + 2 + 1$?

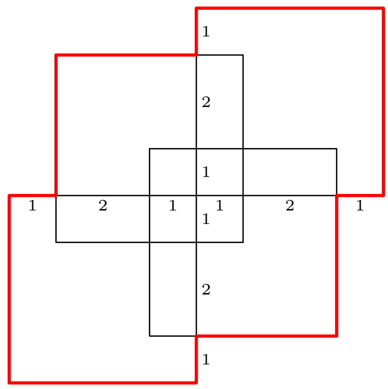
Proposed by: Nathan Xiong

Answer: $\boxed{2067}$

Solution: A quick sum yields

$$2021 + 20 + 21 + 2 + 0 + 2 + 1 = 2021 + 41 + 5 = 2067.$$

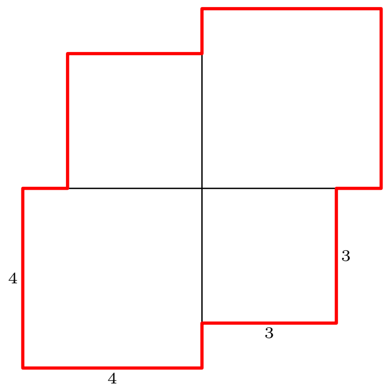
S2. Compute the area of the resulting shape, drawn in red above.



Proposed by: Nathan Xiong

Answer: $\boxed{50}$

Solution: We can divide the shape into four quadrants, as shown below.



Notice that the top-right and bottom-left quadrants each contain a square with side length 4, while the top-left and bottom-right quadrants each contain a square with side length 3. Summing the area across all four quadrants yields an answer of $16 + 16 + 9 + 9 = 50$.

S3. Find the number of ordered pairs (x, y), where x and y are both integers between 1 and 9, inclusive, such that the product $x \times y$ ends in the digit 5.

Proposed by: Andrew Wen

Answer: $\boxed{9}$

Solution: The desired outcome happens if and only if one of x, y is 5, and the other is odd. If $x = 5$, there are 5 options for y: $y = 1, 3, 5, 7, 9$. Similarly, if $y = 5$, there are 5 options for x. However, in these two cases, the pair $(x, y) = (5, 5)$ is counted in both. Hence, the answer is $5 + 5 - 1 = 9$.

S4. Let a, b, and c be real numbers such that $0 \le a, b, c \le 5$ and $2a + b + c = 10$. Over all possible values of a, b, and c, determine the maximum possible value of $a + 2b + 3c$.

Proposed by: Andrew Wen

Answer: $\boxed{25}$

Solution: Note that

$$a + 2b + 3c = (2a + b + c) + (-a + b + 2c) = 10 + (-a + b + 2c).$$

In order to maximize this expression, we need to maximize b and c but minimize a. The optimal way to do this is to let $a = 0$ and $b = c = 5$, in which case the answer is $0 + 10 + 15 = 25$.

S5. There are 12 students in Mr. DoBa's math class. On the final exam, the average score of the top 3 students was 8 more than the average score of the other students, and the average score of the entire class was 85. Compute the average score of the top 3 students.

Proposed by: Yifan Kang

Answer: $\boxed{91}$

Solution: Since the average score of the entire class is 85, the sum of all scores in the class is $85 \cdot 12 = 1020$. Let x be the average score of the top 3 students, so the sum of their scores is $3x$. The average score of the other 9 students is $x - 8$, so the sum of their scores is $9(x - 8)$. Therefore,

$$3x + 9(x - 8) = 1020.$$

Solving for x yields $x = 91$.

S6. Suppose (a, b) is an ordered pair of integers such that the three numbers a, b, and ab form an arithmetic progression, in that order. Find the sum of all possible values of a.

Proposed by: Nathan Xiong

Answer: $\boxed{8}$

Solution: Since a, b, and ab form an arithmetic progression, we have $a + ab = 2b$. Applying Simon's Favorite Factoring Trick, we have

$$ab + a - 2b - 2 = -2 \implies (a - 2)(b + 1) = -2.$$

There are four possibilities to check.

- $a - 2 = -2 \implies a = 0$. Then, $b = 0$, which works.

- $a - 2 = -1 \implies a = 1$. Then, $b = 1$, which works.
- $a - 2 = 1 \implies a = 3$. Then, $b = -3$, which works.
- $a - 2 = 2 \implies a = 4$. Then, $b = -2$, which works.

The answer is $0 + 1 + 3 + 4 = 8$.

S7. If positive real numbers x, y, z satisfy the following system of equations, compute $x + y + z$.

$$xy + yz = 30$$
$$yz + zx = 36$$
$$zx + xy = 42$$

Proposed by: Nathan Xiong

Answer: $\boxed{13}$

Solution: Summing all three equations gives

$$2(xy + yz + zx) = 108 \implies xy + yz + zx = 54.$$

Now, we can subtract each other three equations from this sum to get

$$\begin{cases} zx = (xy + yz + zx) - (xy + yz) = 54 - 30 = 24, \\ xy = (xy + yz + zx) - (yz + zx) = 54 - 36 = 18, \\ yz = (xy + yz + zx) - (zx + xy) = 54 - 42 = 12. \end{cases}$$

Now, we can multiply these three equations to get

$$x^2y^2z^2 = 24 \cdot 18 \cdot 12 \implies xyz = \sqrt{24 \cdot 18 \cdot 12} = 72.$$

Dividing these from our previous products gives

$$\begin{cases} y = \frac{xyz}{zx} = \frac{72}{24} = 3, \\ z = \frac{xyz}{xy} = \frac{72}{18} = 4, \\ x = \frac{xyz}{yz} = \frac{72}{12} = 6. \end{cases}$$

Therefore, $x + y + z = 6 + 3 + 4 = \boxed{13}$.

S8. Andrew chooses three (not necessarily distinct) integers a, b, and c independently and uniformly at random from $\{1, 2, 3, 4, 5, 6, 7\}$. Let p be the probability that $abc(a+b+c)$ is divisible by 4. If p can be written as $\frac{m}{n}$ for relatively prime positive integers m and n, then compute $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{622}$

Solution: If at least two of a, b, c are even, then abc is divisible by 4, so $abc(a+b+c)$ is also divisible by 4. Further, if one of a, b, c is even and the other two are odd, then $a + b + c$ is even, so $abc(a + b + c)$ is divisible by 4. The only case when $abc(a + b + c)$ is not divisible by 4 is when a, b, c are all odd. The probability that this occurs is $\left(\frac{4}{7}\right)^3$. Thus, the desired probability is $1 - \left(\frac{4}{7}\right)^3 = \frac{279}{343}$, and the answer is $279 + 343 = 622$.

S9. Triangle $\triangle ABC$ has $\angle A = 90^\circ$ with $BC = 12$. Square $BCDE$ is drawn such that A is in its interior. The line through A tangent to the circumcircle of $\triangle ABC$ intersects CD and BE at P and Q, respectively. If $PA = 4 \cdot QA$, and the area of $\triangle ABC$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n, then compute $m + n$.

Proposed by: Andy Xu

Answer: $\boxed{149}$

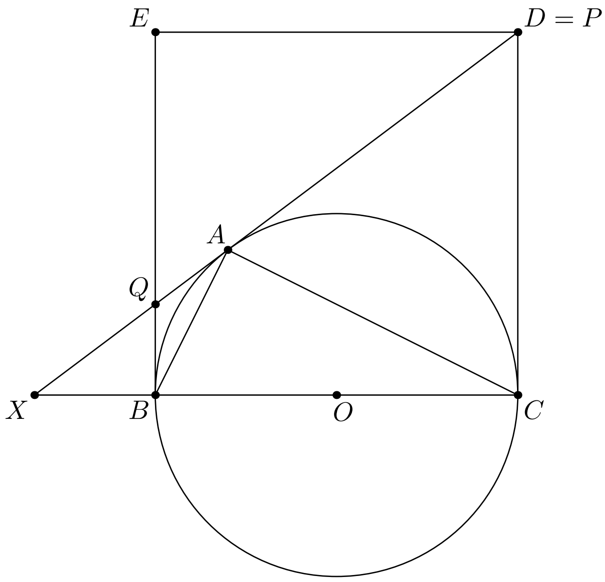
Solution: Let $X = PQ \cap BC$. Since $BQ \parallel PC$, we have $\triangle XBQ \sim \triangle XCP$. Let $QA = x$ and $AP = 4x$. Then $BQ = AQ = x$ and $PA = PC = 4x$. Thus

$$\frac{XB}{XC} = \frac{BQ}{CP} = \frac{1}{4}$$

so $XB = 4$. Similarly

$$\frac{XQ}{XP} = \frac{BQ}{CP} = \frac{1}{4}$$

so $XQ = \frac{5x}{3}$.



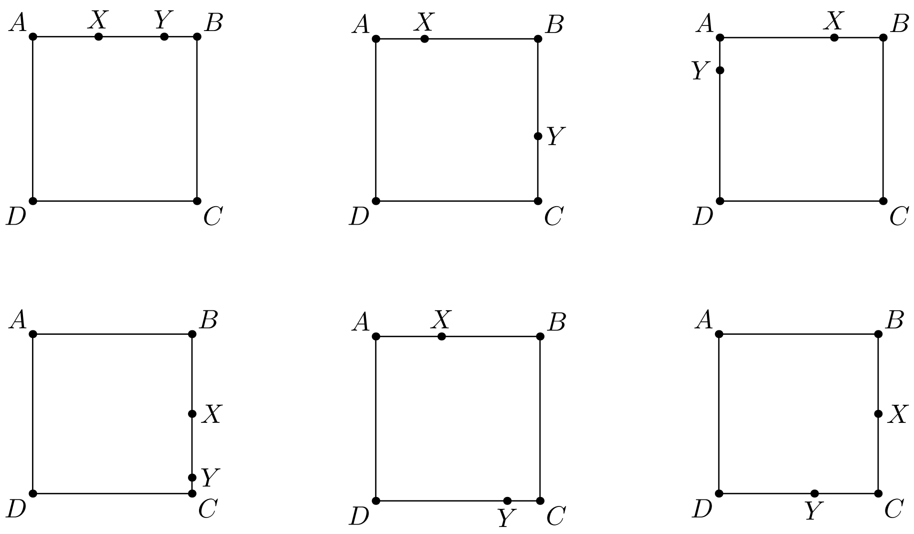
Applying the Pythagorean Theorem on $\triangle QBX$ yields $x = 3$. Next note that $APCO$ is a cyclic quadrilateral because $AO \perp PQ$ and $OC \perp PC$, so $\angle ACB = \angle APO$. Similarly, $\angle ABC = \angle AQO$. Therefore $\triangle OQP \sim \triangle ABC$. We have $QO = 3\sqrt{5}$ and $PO = 6\sqrt{5}$, so $AB = \frac{12}{\sqrt{5}}$ and $AC = \frac{24}{\sqrt{5}}$, so $[ABC] = \frac{144}{5}$, giving an answer of $144 + 5 = 149$.

S10. Let $ABCD$ be a unit square in the plane. Points X and Y are chosen independently and uniformly at random on the perimeter of $ABCD$. If the expected value of the area of triangle $\triangle AXY$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n, compute $m + n$.

Proposed by: Nathan Xiong

Answer: $\boxed{113}$

Solution: We perform casework on what type of configuration we have for which sides X and Y lie on. In total, there are six possible configurations for which sides X and Y lie on, up to symmetry, as shown below. Note that we will assume there are $4^2 = 16$ total possibilities for the sides that X and Y lie on.



- **Top left configuration**: note that this can occur in 2 ways (on side AB or on side AD). In either case, the area of $\triangle AXY$ is always 0.
- **Bottom left configuration**: note that this can occur also in 2 ways. The expected value of the length of XY is $1/3$, since in expectation X and Y split BC into three equal pieces. Therefore, the expected area of $\triangle AXY$ in this case is $\frac{1}{6}$.

For the next four configurations, there's a pretty clean way to argue. First note that the two middle configurations can occur in 4 ways each (can swap X and Y, as well as reflect across AC diagonal), while the two right configurations can only occur in 2 ways each (can only swap X and Y).

Now, note that in any case, if you fix Y at any point on its side, and move X along the side it is fixed to, the area of $\triangle AXY$ will be a linear function in where X is. Therefore, in expectation, the area of $\triangle AXY$ for any given fixed Y will be when X is at the midpoint of its side. Repeat the argument for a fixed X to show that Y must also be at its midpoint. Therefore, the four remaining expected value of areas are

- **Top middle configuration**: 4 ways, with expected area $\frac{1}{8}$.
- **Bottom middle configuration**: 4 ways, with expected area $\frac{1}{4}$.
- **Top right configuration**: 2 ways, with expected area $\frac{1}{8}$.
- **Bottom right configuration**: 2 ways, with expected area $\frac{3}{8}$.

We can summarize all the information we have in a table:

Side	AB	BC	CD	DA
AB	0	1/8	1/4	1/8
BC	1/8	1/6	3/8	1/4
CD	1/4	3/8	1/6	1/8
DA	1/8	1/4	1/8	0

Summing these all up and dividing by 16 gives

$$\frac{2 \cdot 0 + 2 \cdot \frac{1}{6} + 4 \cdot \frac{1}{8} + 4 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} + 2 \cdot \frac{3}{8}}{16} = \frac{17}{96},$$

so the answer is $17 + 96 = 113$.