MOAA 2021 Team Round Solutions

Math Open at Andover

October 16, 2021

T1. The value of

$$\frac{1}{20} - \frac{1}{21} + \frac{1}{20 \times 21}$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by: Nathan Xiong

Answer: $\boxed{211}$

Solution: Compute

$$\frac{1}{20} - \frac{1}{21} + \frac{1}{20 \times 21} = \frac{21 - 20}{20 \times 21} + \frac{1}{20 \times 21} = \frac{2}{20 \times 21} = \frac{1}{210}.$$

The answer is $210 + 1 = 211$.

T2. Four students Alice, Bob, Charlie, and Diana want to arrange themselves in a line such that Alice is at either end of the line, i.e., she is not in between two students. In how many ways can the students do this?

Proposed by: Nathan Xiong

Answer: $\boxed{12}$

Solution: Alice can be at either end of the line, for 2 possible positions. Then, there are always $3! = 6$ ways to arrange the three remaining people, for a total of $2 \times 6 = 12$ possible arrangements.

T3. For two real numbers x and y, let $x \circ y = \frac{xy}{x+y}$. The value of

$$1 \circ (2 \circ (3 \circ (4 \circ 5)))$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by: Nathan Xiong

Answer: $\boxed{197}$

Solution: Note that $x \circ y = \frac{1}{\frac{1}{x} + \frac{1}{y}}$. Therefore, $\frac{1}{x \circ y} = \frac{1}{x} + \frac{1}{y}$. Use this definition to expand:

$$\frac{1}{1 \circ (2 \circ (3 \circ (4 \circ 5)))} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \implies 1 \circ (2 \circ (3 \circ (4 \circ 5))) = \frac{60}{137}.$$

The answer is $60 + 137 = 197$.

T4. Compute the number of ordered triples (x, y, z) of integers satisfying

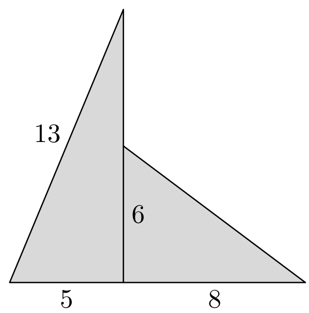
$$x^2 + y^2 + z^2 = 9.$$

Proposed by: Nathan Xiong

Answer: $\boxed{30}$

Solution: Ignoring order, there are two possible triples of perfect squares that sum to 9: $1+4+4 = 0+0+9 = 9$. In the first case, accounting for permutations and negative numbers, there are $3 \times 2^3 = 24$ valid ordered triples. In the second case, there are $3 \times 2 = 6$ triples. The answer is $24 + 6 = 30$.

T5. Two right triangles are placed next to each other to form a quadrilateral as shown. What is the perimeter of the quadrilateral?



Proposed by: Nathan Xiong

Answer: $\boxed{42}$

Solution: By the Pythagorean Theorem, the hypotenuse of the right triangle is 10. Also by the Pythagorean Theorem, the missing height of the left triangle is 12. So, the perimeter is just

$$13 + 5 + 8 + 10 + (12 - 6) = 42.$$

T6. Find the sum of all two-digit prime numbers whose digits are also both prime numbers.

Proposed by: Nathan Xiong

Answer: $\boxed{186}$

Solution: The digits of the prime number must be in the set $\{2, 3, 5, 7\}$. Next, the prime number clearly cannot end in 2 or 5, so it must end in 3 or 7. Checking all 8 possibilities gives us 4 valid primes: $23, 37, 53, 73$. The answer is the sum of these four numbers, which is 186.

T7. Compute the number of ordered pairs (a, b) of positive integers satisfying $a^b = 2^{100}$.

Proposed by: Nathan Xiong

Answer: $\boxed{9}$

Solution: Taking the b-th root of both sides yields

$$a = 2^{\frac{100}{b}}.$$

For a to be a positive integer, $\frac{100}{b}$ must be a positive integer. Note that $100 = 2^2 \times 5^2$ has 9 factors, and each value of b corresponds to a singular value of a. Hence, the answer is just 9.

T8. Evaluate

$$2^7 \times 3^0 + 2^6 \times 3^1 + 2^5 \times 3^2 + \cdots + 2^0 \times 3^7.$$

Proposed by: Nathan Xiong

Answer: $\boxed{6305}$

Solution: Let $x = 2$ and $y = 3$. The expression is equivalent to

$$x^7 + x^6y^1 + \cdots + x^1y^6 + y^7 = \frac{y^8 - x^8}{y - x}.$$

This is equal to $3^8 - 2^8 = 6561 - 256 = 6305$. The expression can also be computed by interpreting it as a geometric series with common ratio $\frac{3}{2}$.

T9. Mr. DoBa has a bag of markers. There are 2 blue, 3 red, 4 green, and 5 yellow markers. Mr. DoBa randomly takes out two markers from the bag. The probability that these two markers are different colors can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by: Raina Yang

Answer: $\boxed{162}$

Solution: We calculate the probability that the two drawn marbles have the same color. This is

$$P = \frac{\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2}}{\binom{14}{2}} = \frac{1 + 3 + 6 + 10}{91} = \frac{20}{91}.$$

So, our desired probability is $1 - P = \frac{71}{91}$, and the answer is $71 + 91 = 162$.

T10. For how many nonempty subsets $S \subseteq \{1, 2, \ldots, 10\}$ is the sum of all elements in S even?

Proposed by: Andrew Wen

Answer: $\boxed{511}$

Solution: Note that the sum of all elements in $\{1, 2, \ldots, 10\}$ is $1+2+\cdots+10 = 55$, which is odd. So, for every subset S, the sum of the elements in the complement of S has the opposite parity of S. Hence, exactly half of all $2^{10} = 1024$ subsets have even sum, counting the empty set. Therefore, excluding the empty set, the desired answer is $512 - 1 = 511$.

T11. Find the product of all possible real values for k such that the system of equations

$$\begin{cases} x^2 + y^2 = 80 \\ x^2 + y^2 = k + 2x - 8y \end{cases}$$

has exactly one real solution (x, y).

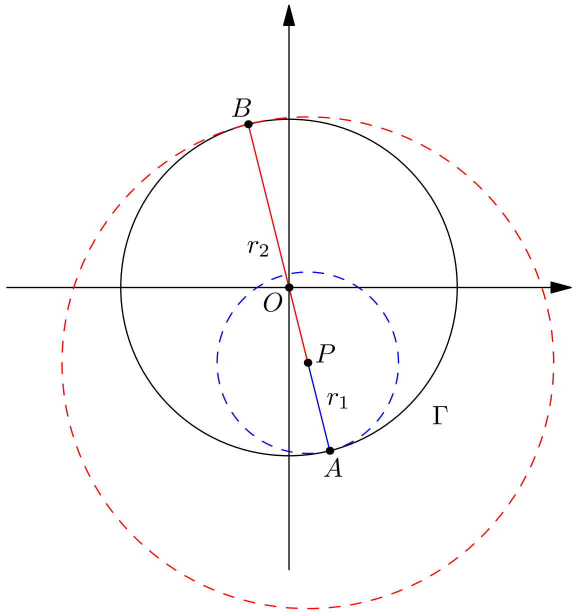
Proposed by: Nathan Xiong

Answer: $\boxed{960}$

Solution: The key is to note that both of these equations are circles in the xy-plane. The first is simply a circle Γ centered at the origin with radius $\sqrt{80}$, and by rearranging the second equation and completing the square, we can obtain

$$x^2 - 2x + 1 + y^2 + 8y + 16 = k + 17 \implies (x-1)^2 + (y+4)^2 = k + 17,$$

which is a circle ω centered at $(1, -4)$ with radius $r = \sqrt{k + 17}$. Now, note that ω is only tangent to Γ in two possible cases:



Note that the two possible radii are labelled r_1 and r_2 above. Label the other points as above as well. By power of a point, we have that

$$r_1 \cdot r_2 = \text{power of } P \text{ with respect to } \Gamma = R^2 - OP^2 = 80 - 17 = 63,$$

where R is the radius of Γ. In addition, note that $r_1 + r_2 = 2\sqrt{80} = 8\sqrt{5}$.

Now, we want to compute the product of all possible k. Note that the two possible values of k are $k_1 = r_1^2 - 17$ and $k_2 = r_2^2 - 17$. Then,

$$k_1k_2 = (r_1^2 - 17)(r_2^2 - 17) = r_1^2r_2^2 - 17(r_1^2 + r_2^2) + 289.$$

From our identities above about the sum and product of r_1 and r_2, this evaluates to

$$63^2 - 17((8\sqrt{5})^2 - 2 \cdot 63) + 289 = 960.$$

T12. Let $\triangle ABC$ have $AB = 9$ and $AC = 10$. A semicircle is inscribed in $\triangle ABC$ with its center on segment BC such that it is tangent AB at point D and AC at point E. If $AD = 2DB$ and r is the radius of the semicircle, r^2 can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by: Andy Xu

Answer: $\boxed{415}$

Solution: If O is the circumcircle, notice that $\triangle ADO \cong \triangle AEO$, so AO is the angle bisector of $\angle A$. We have $AD = AE = 6$, so $BD = 3$ and $EC = 4$. By the angle bisector theorem,

$$\frac{AB}{BO} = \frac{AC}{CO} \implies \frac{9}{\sqrt{r^2+9}} = \frac{10}{\sqrt{r^2+16}}$$

which yields $r^2 = \frac{396}{19}$. The answer is $396 + 19 = 415$.

T13. Bob has 30 identical unit cubes. He can join two cubes together by gluing a face on one cube to a face on the other cube. He must join all the cubes together into one connected solid. Over all possible solids that Bob can build, what is the largest possible surface area of the solid?

Proposed by: Nathan Xiong

Answer: $\boxed{122}$

Solution: The maximum is attained if we just create a huge line of 30 cubes. To prove this, note that every time we attach another cube to our solid, the surface area increased by at most 4. This implies that the largest possible surface area is $6 + 29 \times 4 = 122$. A single line of 30 cubes indeed achieves this maximum.

T14. Evaluate

$$\left\lfloor \frac{1 \times 5}{7} \right\rfloor + \left\lfloor \frac{2 \times 5}{7} \right\rfloor + \left\lfloor \frac{3 \times 5}{7} \right\rfloor + \cdots + \left\lfloor \frac{100 \times 5}{7} \right\rfloor.$$

Note: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Proposed by: Nathan Xiong

Answer: $\boxed{3564}$

Solution: Consider the numbers seven numbers $1 \times 5, 2 \times 5, \ldots, 7 \times 5$ under (mod 7). Since $\gcd(5, 7) = 1$, these seven numbers are just some permutation of $0, 1, \ldots, 6$. Then, let $r(n)$ denote the remainder when n is divided by 7. Clearly, for any integer n

$$\left\lfloor \frac{n}{7} \right\rfloor = \frac{n}{7} - \frac{r(n)}{7}.$$

So, our sum is

$$\begin{aligned}
\sum_{i=1}^{100} \left\lfloor \frac{5i}{7} \right\rfloor &= \sum_{i=1}^{100} \left(\frac{5i}{7} - \frac{r(5i)}{7} \right), \\
&= \frac{5}{7}(1 + \cdots + 100) - \frac{14}{7}(0 + 1 + \cdots + 6) - \frac{r(99 \times 5)}{7} - \frac{r(100 \times 5)}{7}, \\
&= \frac{5 \times 5050}{7} - 2 \times 21 - \frac{5}{7} - \frac{3}{7}, \\
&= 3606 - 42, \\
&= 3564.
\end{aligned}$$

T15. Consider the polynomial

$$P(x) = x^3 + 3x^2 + 6x + 10.$$

Let its three roots be a, b, c. Define $Q(x)$ to be the monic cubic polynomial with roots ab, bc, ca. Compute $|Q(1)|$.

Proposed by: Nathan Xiong

Answer: $\boxed{75}$

Solution: By the definition of Q,

$$Q(1) = (1 - ab)(1 - bc)(1 - ca) = \frac{1}{abc}(a - abc)(b - abc)(c - abc).$$

By Vieta's formulas, we have $abc = -10$. Now,

$$P(-10) = P(abc) = (abc - a)(abc - b)(abc - c) = -750.$$

Finally,

$$Q(1) = -\frac{1}{-10}(-750) = -75,$$

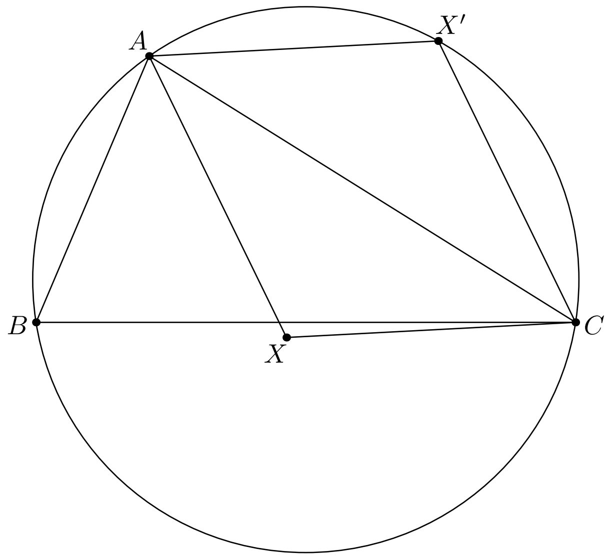
and the answer is 75.

T16. Let $\triangle ABC$ have $\angle ABC = 67^\circ$. Point X is chosen such that $AB = XC$, $\angle XAC = 32^\circ$, and $\angle XCA = 35^\circ$. Compute $\angle BAC$ in degrees.

Proposed by: Raina Yang

Answer: $\boxed{81}$

Solution: Let X' be the point such that $AXCX'$ is a parallelogram.



Then, $\angle AX'C = \angle AXC = 180^\circ - 32^\circ - 35^\circ = 113^\circ$. So, $ABCX'$ is a cyclic quadrilateral. Furthermore, since $AB = XC = X'A$, we have $\angle ABX' = \angle AX'B$. Since $AXCX'$ is a parallelogram, $\angle ABX' = \angle ACX' = \angle CAX = 32^\circ$. Finally, we have

$$\angle BAC = \angle BX'C = \angle AX'C - \angle AX'B = 113^\circ - \angle ABX' = 113^\circ - 32^\circ = 81^\circ.$$

T17. Compute the remainder when 10^{2021} is divided by 10101.

Proposed by: Nathan Xiong

Answer: $\boxed{9091}$

Solution: Note that

$$10^6 - 1 = (10^2 - 1)(10^4 + 10^2 + 1) = 99 \times 10101.$$

In particular, $10101 \mid 10^6 - 1 \implies 10^6 \equiv 1 \pmod{10101}$. Hence,

$$10^{2021} \equiv 10^5 \pmod{10101},$$

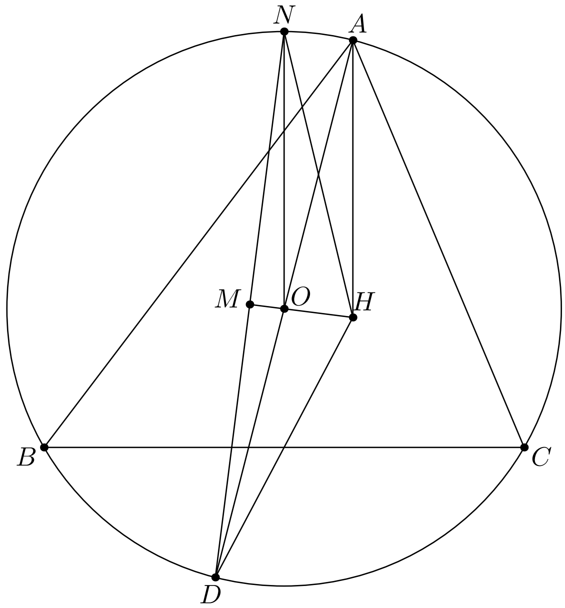
which can be calculated to be $100000 - 9 \times 10101 = 9091$.

T18. Let $\triangle ABC$ be a triangle with side length $BC = 4\sqrt{6}$. Denote ω as the circumcircle of $\triangle ABC$. Point D lies on ω such that AD is the diameter of ω. Let N be the midpoint of arc BC that contains A. H is the intersection of the altitudes in $\triangle ABC$ and it is given that $HN = HD = 6$. If the area of $\triangle ABC$ can be expressed as $\frac{a\sqrt{b}}{c}$, where a, b, c are positive integers with a and c relatively prime and b not divisible by the square of any prime, compute $a + b + c$.

Proposed by: Andy Xu

Answer: $\boxed{52}$

Solution: Let M be the midpoint of ND. Since $\triangle HND$ is isosceles, we have that $HM \perp ND$. We also have $OM \perp ND$ so H, O, M are collinear.



We now prove that O is the centroid of $\triangle HND$. It is well known that $P = HD \cap BC$ is the midpoint of BC. We are given that N is the midpoint of arc BC, so we know that N, O, P are collinear. Note that this implies that O is the centroid of $\triangle HND$ because H, O, M are collinear.

Let the circumradius of $\triangle ABC$ be R. Since $NO = 2OP$, we have $OP = \frac{R}{2}$. Note that $OB = R$, so $\triangle OBP$ is a 30-60-90 triangle. Therefore, $\angle A = 60°$. Using the fact that $BC = 4\sqrt{6}$ we find that $R = \frac{BC}{\sqrt{3}} = 4\sqrt{2}$.

We now prove that $AHON$ is a parallelogram. We have $DO \cap HN = Q$ is the midpoint of HN. Since $OD = R$, $OQ = \frac{R}{2}$, and $AO = R$, we see that Q bisects AO. Thus $AHON$ is a parallelogram.

Let $OM = x$, so that $OH = 2x$. Then, Pythagoras theorem $HN^2 - HM^2 = NO^2 - OM^2$ yields $x = \frac{\sqrt{2}}{2}$. It follows that $OH = \sqrt{2}$.

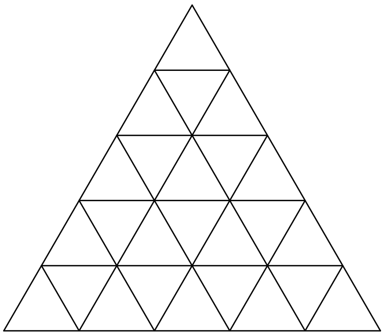
Using the fact that $AHON$ is a parallelogram, we have $AH = ON = R = 4\sqrt{2}$. Let line AH intersect BC at X and ω again at $Y \neq A$. It is well known that Y is the reflection of H about X, so $HX = XY$.

By Power of a Point, we have $R^2 - OH^2 = AH \cdot HY = AH \cdot 2HX$ which yields that $HX = \frac{15}{4\sqrt{2}}$. Thus

$$AX = AH + HX = \frac{47}{4\sqrt{2}}.$$

Finally, we find $[ABC] = \frac{AD \cdot BC}{2} = \frac{47\sqrt{3}}{2}$ which means the answer is $47 + 3 + 2 = 52$.

T19. Consider the 5 by 5 by 5 equilateral triangular grid as shown:



Ethan chooses two distinct upward-oriented equilateral triangles bounded by the gridlines. The probability that Ethan chooses two triangles that share exactly one vertex can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{23}$

Solution: We assign coordinates (x, y, z) to each point, where x is the distance to the bottom side, y is the distance to the left side, and z is the distance to the right side, assuming that the height of one small triangle is 1. Note that we always have $x + y + z = 10$. Then, if the two triangles meet at point $P = (x, y, z)$ and only at P, then the number of ways to choose these two triangles is

$$\binom{x}{2} + \binom{y}{2} + \binom{z}{2} + xy + yz + zx = \frac{1}{2}((x + y + z)^2 - (x + y + z)) = 10.$$

Multiplying across all 21 points in the triangular grid yields 210 ways to choose two valid triangles. There are a total of $15 + 10 + 6 + 3 + 1 = 35$ triangles in the grid, so the desired probability is

$$\frac{210}{\binom{35}{2}} = \frac{6}{17}.$$

The answer is $6 + 17 = 23$.

T20. Compute the sum of all integers x for which there exists an integer y such that

$$x^3 + xy + y^3 = 503.$$

Proposed by: Nathan Xiong

Answer: $\boxed{21}$

Solution: Instead of using x, y, define the new variables m, n by $m = -3x, n = -3y$. Note that m and n are both multiples of 3. Substituting into the equation above and simplifying, we get

$$\left(-\frac{m}{3}\right)^3 + \left(-\frac{m}{3}\right)\left(-\frac{n}{3}\right) + \left(-\frac{n}{3}\right)^3 = 503 \implies m^3 - 3mn + n^3 = -13581.$$

We can apply the "well-known" factorization for $a^3 + b^3 + c^3 - 3abc$ now by adding a 1 to both sides.

$$\begin{aligned} m^3 - 3mn + n^3 + 1 &= -13580 \\ (m + n + 1)(m^2 + n^2 + 1 - mn - m - n) &= -13580. \end{aligned}$$

We'll make several simplifications before writing out all possibilities. First, since m, n are multiples of 3, both terms in the product on the left hand side are 1 (mod 3). Furthermore, it's easy to see (say, by casework on the signs of m, n) that the second term in the product is always positive. Hence, the first term in the product is always positive. Furthermore, it's easy to see that the magnitude of the second term is also larger than the magnitude of the first term.

Now, we can calculate the prime factorization $13580 = 2^2 \times 5 \times 7 \times 97$. Consider the first term in the product. It must be negative, 1 (mod 3), and have magnitude less than $\sqrt{13580} \approx 116$. By listing out all the divisors of 13580 and checking, there are only five possible cases for the value of $m + n + 1$: $-2, -5, -14, -20, -35$. Finally, we check all five cases.

Case 1: $m + n + 1 = -2$. Then, we have

$$m^2 + n^2 + 1 - mn - m - n = 6790.$$

By squaring the first equation and cancelling out terms with the second equation, we can end up at $mn = -2259$. However, since $2259 = 3^2 \times 251$, there are no valid m, n in this case.

Case 2: $m + n + 1 = -5$. Then, we have

$$m^2 + n^2 + 1 - mn - m - n = 2716.$$

Similarly to the previous case, through routine equation manipulation, we can end up at $mn = -891$. This has integer solutions $(m, n) = (-33, 27), (27, -33)$.

Case 3: $m + n + 1 = -14$. Then, we have

$$m^2 + n^2 + 1 - mn - m - n = 970.$$

Through routine equation manipulation, we can end up at $mn = -243$. However, since $243 = 3^5$, there are no valid m, n in this case.

Case 4: $m + n + 1 = -20$. Then, we have

$$m^2 + n^2 + 1 - mn - m - n = 679.$$

Through routine equation manipulation, we can end up at $mn = -72$. This has integer solutions $(m, n) = (-24, 3), (3, -24)$.

Case 5: $m + n + 1 = -35$. Then, we have

$$m^2 + n^2 + 1 - mn - m - n = 388.$$

Through routine equation manipulation, we can end up at $mn = 315$. This has integer solutions $(m, n) = (-21, -15), (-15, -21)$.

Finally, converting back into variables x, y, we get that there are six total solutions: $(x, y) = (11, -9), (-9, 11), (8, -1), (-1, 8), (7, 5), (5, 7)$. And the sum of all possible x values is $11 - 9 + 8 - 1 + 7 + 5 = 21$.