MOAA 2022 Speed Round Solutions

Math Open at Andover

Ocotber 8th, 2022

S1. What is the value of the sum

$$2 + 20 + 202 + 2022?$$

Proposed by: Andy Xu

Answer: $\boxed{2246}$

Solution: Simply adding yields

$$\begin{aligned} 2 + 20 + 202 + 2022 &= 22 + 202 + 2022 \\ &= 224 + 2022 \\ &= \boxed{2246} \end{aligned}$$

as desired.

S2. Find the smallest integer greater than 10000 that is divisible by 12.

Proposed by: Andy Xu

Answer: $\boxed{10008}$

Solution: Check that 10000 divided by 12 (via long division) is 833 with remainder 4. Hence, it follows that the least number k to be added to 10000 to arrive at a multiple of 12 is $k = 8$, so our answer is $\boxed{10008}$.

S3. Andy chooses a positive integer factor of 6^{10} at random. The probability that it is odd can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{12}$

Solution: Every factor is of the form $2^a 3^b$, where a, b range from 0 to 10. Since each of a, b has 11 choices, there are a total of $11^2 = 121$ factors. A factor is odd if $a = 0$, of which there are 11 choices for b, so there are 11 odd factors. Our answer is $\frac{11}{121} = \frac{1}{11} \implies \boxed{12}$.

S4. How many three digit positive integers are multiples of 4 but not 8?

Proposed by: Andrew Wen

Answer: $\boxed{113}$

Solution: We observe there are $\lfloor \frac{999}{4} \rfloor - \lfloor \frac{99}{4} \rfloor = 225$ multiples of 4 less than 1000, exactly $\lfloor \frac{999}{8} \rfloor - \lfloor \frac{99}{8} \rfloor = 112$ of which are multiples of 8.

Therefore, the answer is $225 - 112 = \boxed{113}$.

S5. At Wen's Wontons, Andy accidentally eats 5 dollars more worth of Wontons than he had planned. Originally, including the tip, he planned to spend all of the remaining 90 dollars on his giftcard. To compensate for his gluttony, Andy instead gives the waiter a smaller, 12.5% tip so that he still spends 90 dollars total. How much percent tip was Andy originally planning on giving?

Proposed by: Andy Xu

Answer: $\boxed{20}$

Solution: If M is the amount of money Andy actually spends, then

$$M + 0.125M = 90 \implies M = 80.$$

Hence, he intended to spend 75 dollars, which would entail a tip of 15 dollars. This is 20% of 75 dollars, so our answer is $\boxed{20}$.

S6. Let A, B, C, D be four coplanar points satisfying the conditions $AB = 16$, $AC = BC = 10$, and $AD = BD = 17$. What is the minimum possible area of quadrilateral $ADBC$?

Proposed by: Andrew Wen

Answer: $\boxed{72}$

The length conditions tell us that C, D lie on the perpendicular bisector of segment AB. Therefore, to minimize $[ACBD]$, clearly we want C, D to be on the same side as segment AB, which means $ACBD$ is concave.

If M is the midpoint of AB, then $CM = \sqrt{10^2 - 8^2} = 6$ and $DM = \sqrt{17^2 - 8^2} = 15$. Then,

$$[ACBD] = [DAB] - [CAB] = \tfrac{1}{2}(16 \cdot 15) - \tfrac{1}{2}(16 \cdot 6) = \boxed{72}$$

as desired.

S7. How many ways are there to select a set of three distinct points from the vertices of a regular hexagon so that the triangle they form has its smallest angle(s) equal to 30°?

Proposed by: Andy Xu

Answer: $\boxed{18}$

Solution: There are three kinds of triangles we can form: either $30-30-120$, $30-60-90$, or $60-60-60$. There are a total of $\binom{6}{3} = 20$ ways of choosing three distinct points, of which there are 2 distinct equilateral triangles. So, our answer is therefore $20 - 2 = \boxed{18}$.

S8. Phillip rolls five fair 6-sided die. The probability that the sum of some three rolls is exactly 8 times the sum of the other two rolls can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{653}$

Solution: The maximum possible roll sum of three rolls is $6 + 6 + 6 = 18$. The minimum possible roll sum of two rolls is $1 + 1 = 2$. If the sum of the other two rolls is ≥ 3, then the sum of the three rolls must be ≥ 24, impossible, so the sum of some three rolls must be exactly 16, while the sum of the other two rolls is 2, formed by exactly two rolls of 1. Because 16 can only be written as $6 + 6 + 4$ and $6 + 5 + 5$, there are two cases:

- Rolls of $6, 6, 4, 1, 1$ in some order. There are $\frac{5!}{2!\cdot 2!} = 30$ permutations of the rolls, where each possible roll has a $\frac{1}{6^5}$ chance of happening, so this case overall has a probability of $\frac{30}{6^5}$.
- Rolls of $6, 5, 5, 1, 1$ in some order. There are $\frac{5!}{2!\cdot 2!} = 30$ permutations of the rolls, where each possible roll has a $\frac{1}{6^5}$ chance of happening, so this case overall has a probability of $\frac{30}{6^5}$.

Summing yield a probability of $\frac{60}{6^5} = \frac{5}{648} \implies \boxed{653}$.

S9. Find the least positive integer n for there exists some positive integer $k > 1$ for which k and $k + 2$ both divide $\underbrace{11\ldots1}_{n \text{ 1's}}$.

Proposed by: Andrew Wen

Answer: $\boxed{6}$

Solution: Abuse the fact that $11, 13 \mid 1001$. It follows that $n = \boxed{6}$ works. It should be clear that $n \leq 6$ do not work, as $1 = 1, 11 = 11, 111 = 3 \times 37, 1111 = 11 \times 101$, and with a little bit of work, $11111 = 41 \times 271$.

S10. For some real constant k, line $y = k$ intersects the curve $y = |x^4 - 1|$ four times: points A, B, C and D, labeled from left to right. If $BC = 2AB = 2CD$, then the value of k can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{32}$

Solution: The function $|x^4 - 1|$, so its graph is symmetrical about the y-axis. Then, the midpoint of BC is just $(0, k)$. Label it to be O. The condition translates to $2OC = OD$.

From analyzing the graph of $|x^4 - 1|$, which is just $y = x^4 - 1$ but with the part below the x-axis reflected above (by the absolute value), note that the length of OD is the absolute value of the solution(s) to $x^4 - 1 = k$, and similarly the length of OC is just the absolute value of the solutions(s) to $1 - x^4 = k$. Hence,

$$OD = 2OC \iff \sqrt[4]{1 + k} = 2\sqrt[4]{1 - k}$$

so solving for k, we get $1 + k = 16(1 - k)$ so $k = \frac{15}{17} \implies 15 + 17 = \boxed{32}$.

S11. Let a be a positive real number and $P(x) = x^2 - 8x + a$ and $Q(x) = x^2 - 8x + a + 1$ be quadratics with real roots such that the positive difference of the roots of $P(x)$ is exactly one more than the positive difference of the roots of $Q(x)$. The value of a can be written as a common fraction $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{247}$

Solution: Note that in a quadratic $x^2 - px + q$, if its roots are r and s, then we have that $r + s = p, rs = q \implies |r - s| = \sqrt{p^2 - 4q}$ where $p^2 - 4q \geq 0$ if r, s are real. Hence, the problem condition actually translates to

$$\sqrt{64 - 4a} - \sqrt{64 - 4(a + 1)} = 1 \implies \sqrt{16 - a} - \sqrt{15 - a} = \tfrac{1}{2}.$$

Note that

$$1 = (\sqrt{16-a}+\sqrt{15-a})(\sqrt{16-a}-\sqrt{15-a}) = \tfrac{1}{2}(\sqrt{16-a}-\sqrt{15-a})$$

so it follows that we have the equation $\sqrt{16-a}+\sqrt{15-a}=2$. Adding $\sqrt{16-a}-\sqrt{15-a}=\frac{1}{2}$ gives

$$2\sqrt{16-a}=\tfrac{5}{2} \implies 16-a=\tfrac{25}{16} \implies a=\tfrac{231}{16}$$

for an answer of $231+16=\boxed{247}$.

S12. Let $ABCD$ be a trapezoid satisfying $AB \parallel CD$, $AB=3, CD=4$, with area 35. Given AC and BD intersect at E, and M,N,P,Q are the midpoints of segments AE,BE,CE,DE, respectively, the area of the intersection of quadrilaterals $ABPQ$ and $CDMN$ can be expressed as $\frac{m}{n}$ where m,n are relatively prime positive integers. Find $m+n$.

Proposed by: Andrew Wen

Answer: $\boxed{313}$

Solution: Let $BP\cap CN=X$ and $AQ\cap DM=Y$. We want to find the area of $MNXPQY$. If we let $[ABE]=3A$, then by area ratios,

$$\frac{[ADE]}{[ABE]}=\frac{DE}{BE}=\frac{CD}{AB}=\frac{4}{3} \implies [ADE]=4A$$

and similarly $[BCE]=4A$. Finally, note $[CDE]/[ABE]=(CD/AB)^2=\frac{16}{9}$ so it follows that $[CDE]=\frac{16}{3}A$. So

$$35=[ABCD]=3A+4A+4A+\tfrac{16}{3}A \implies A=\tfrac{15}{7}.$$

Finally, since by definition X and Y are centroids of BCE and ADE, it follows that

$$\begin{aligned}[MNXPQY] &= [MNE]+[NXPE]+[PQE]+[QYME]\\ &= \tfrac{1}{4}[ABE]+\tfrac{1}{3}[BCE]+\tfrac{1}{4}[CDE]+\tfrac{1}{3}[DAE]\\ &= \tfrac{3A}{4}+\tfrac{4A}{3}+\tfrac{4A}{3}+\tfrac{4A}{3}\\ &= \tfrac{19A}{4}\\ &= \tfrac{285}{28}\end{aligned}$$

for a final answer of $285+28=\boxed{313}$.

S13. There are 8 distinct points $P_1, P_2, \ldots, P_8$ on a circle. How many ways are there to choose a set of three distinct chords such that every chord has to touch at least one other chord, and if any two chosen chords touch, they must touch at a shared endpoint?

Proposed by: Andrew Wen

Answer: $\boxed{896}$

Solution: There are three cases:

- If the three chords all share one vertex, then there are 8 ways to select that vertex, and $\binom{7}{3}$ ways to choose the three chords emanating from that vertex, for a total of $8\times 35=280$ triples.

- If the three chosen chords form a triangle, there are $\binom{8}{3} = 56$ ways of choosing a triangle among the 8 points.
- If the three chosen chords do not form a triangle or all share a vertex, then we can cover all such set choices by first choosing a closed loop of length 4, then deleting one of the 4 edges of the loop so that no two of the remaining 3 cross in the interior. Note that each set of four points defines 3 loops, a regular quadrilateral and two self-intersecting ones. There are 4 ways to delete from the regular one, and 2 ways to delete from each of the self-intersecting ones (we have to delete one of the two self-intersecting edges) so our final count is

$$(4 + 2 \times 2) \times \binom{8}{4} = 560$$

  in this case.

Our final answer is $560 + 56 + 280 = \boxed{896}$.

S14. For every positive integer k, let $f(k) > 1$ be defined as the smallest positive integer for which $f(k)$ and $f(k)^2$ leave the same remainder when divided by k. The minimum possible value of $\frac{1}{x}f(x)$ across all positive integers $x \leq 1000$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Find $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{32}$

Solution: We claim the minimum is $\frac{1}{31}$, achieved when $k = 32 \times 31 \implies f(k) = 32$.

So, $f(k)$ is the smallest positive integer $n \geq 2$ for which $k \mid n(n-1)$. In order to construct n, there is a splitting of k into $A \cdot B$ where A, B are relatively prime and $A \mid n, B \mid n - 1$. Thus, $n = Am_1, Bm_2 + 1$ for positive integers m_1, m_2. Here,

$$\frac{f(k)}{k} = \frac{n}{AB} = \sqrt{\frac{(Am_1)(Bm_2 + 1)}{A^2B^2}} > \sqrt{\frac{m_1m_2}{k}}.$$

If any one of $m_1, m_2 > 1$, then $\frac{f(k)}{k} \geq \sqrt{\frac{2}{k}} \geq \sqrt{\frac{1}{500}} > \frac{1}{31}$.

Otherwise, if both $m_1 = m_2 = 1$, then $n = A = B + 1$, so k can be expressed as $AB = n(n-1)$. Then, we would have

$$\frac{f(k)}{k} = \frac{n}{n(n-1)} = \frac{1}{n-1} \geq \frac{1}{31}$$

where $n \leq 32$ since $k = n(n-1) \leq 1000$. Indeed, $\frac{1}{k}f(k) \geq \frac{1}{31}$ in all cases, as desired.

So, indeed, our answer is $\frac{1}{31} \implies \boxed{32}$.

S15. In triangle ABC, let I be the incenter and O be the circumcenter. If AO bisects $\angle IAC$, $AB + AC = 21$, and $BC = 7$, then the length of segment AI can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Andy Xu

Answer: $\boxed{143}$

Solution: Let AI intersect BC at D and the circumcircle again at M, midpoint of small arc BC. First, note that:

Claim: Triangle ABD is isosceles with $AB = AD$.

Proof: If $\angle BAC = 4\theta$ then $\angle BAO = \angle ABO = 3\theta$ and $\angle CAO = \angle ACO = \theta$, and if we let $\angle BCO = \angle CBO = \gamma$, then

$$\angle ABD = 3\theta + \gamma = 2\theta + (\theta + \gamma) = \angle DAC + \angle ACD = \angle ADB$$

which proves the desired claim. □

Now, we let $AI = \ell$ and length chase. Note that by Angle Bisector Theorem

$$\frac{AI}{ID} = \frac{BA}{BD} = \frac{CA}{CD} = \frac{AB + AC}{BD + CD} = \frac{21}{7} = 3$$

so $ID = \frac{\ell}{3} \implies AD = AB = \frac{4\ell}{3}$. Next, since $MI = MB = MC = d$ by the well known Incenter-Excenter Lemma, note that by Ptolemy on $ABMC$,

$$d(AB + AC) = 7(\ell + d) \implies 3d = \ell + d \implies d = \tfrac{\ell}{2}.$$

So, $AM = \ell + d = \frac{3}{2}\ell$. Finally, note from Power of a Point at D, we get

$$BD \cdot CD = \left(7 \cdot \frac{\frac{4}{3}\ell}{21}\right)\left(7 \cdot \frac{21 - \frac{4}{3}\ell}{21}\right) = AD \cdot MD = \left(\tfrac{4}{3}\ell\right)\left(\tfrac{3}{2}\ell - \tfrac{4}{3}\ell\right)$$

which upon simplification, yields

$$\frac{4}{9}\ell\left(7 - \frac{4}{9}\ell\right) = \frac{4}{3}\ell \cdot \frac{1}{6}\ell \implies 7 - \frac{4}{9}\ell = \frac{1}{2}\ell$$

so in fact $\ell = \frac{126}{17} \implies 126 + 17 = \boxed{143}$.