MOAA 2022 Accuracy Round Solutions

Math Open at Andover

October 8, 2022

A1. Find the last digit of 2022^{2022}.

Proposed by: Yifan Kang

Answer: $\boxed{4}$

Solution: Note that $2022^{2022} \equiv 2^{2022} \pmod{10}$. Furthermore, $2^5 \equiv 2^1 \pmod{10}$ so $2^x \pmod{10}$ cycles every 4: that is 2^x and 2^{x+4} always have the same units digit. Hence

$$2^{2022} \equiv 2^{2022-4\times 500} \equiv 2^2 \equiv 4 \pmod{10}$$

so our final answer is $\boxed{4}$.

A2. Let $a_1 < a_2 < \ldots < a_8$ be eight real numbers in an increasing arithmetic progression. If $a_1 + a_3 + a_5 + a_7 = 39$ and $a_2 + a_4 + a_6 + a_8 = 40$, determine the value of a_1.

Proposed by: Andrew Wen

Answer: $\boxed{9}$

Solution: Subtracting the first equation from the second, we get

$$(a_2 - a_1) + (a_4 - a_3) + (a_6 - a_5) + (a_8 - a_7) = 4d = 1$$

where d is the common difference of the sequence. Now note that

$$\begin{aligned} a_1 + a_3 + a_5 + a_7 &= a_1 + (a_1 + 2d) + (a_1 + 4d) + (a_1 + 6d) \\ &= 4a_1 + 12d \\ &= 39 \end{aligned}$$

which tells us $4a_1 = 39 - 12d = 36$ so our final answer is $a_1 = \boxed{9}$.

A3. Lucas tries to evaluate the sum of the first 2022 positive integers, but accidentally omits one of the numbers, N, while adding all of them manually, and incorrectly arrives at a multiple of 1000. Assuming that he adds correctly otherwise, find the sum of all possible values of N.

Proposed by: Andrew Wen

Answer: $\boxed{1506}$

Solution: If N is the number he omits, then his sum is

$$(1 + ... + 2022) - N = 1011 \times 2023 - N \equiv 11 \times 23 - N \equiv 0 \pmod{1000}.$$

This tell us $N \equiv 23 \times 11 = 253 \pmod{1000}$, and since we know $1 \leq N \leq 2022$ it follows $N = 253, 1253$ so our answer is $253 + 1253 = \boxed{1506}$.

A4. A machine picks a real number uniformly at random from$[0, 2022]$. Jack randomly chooses a real number from $[2020, 2022]$. The probability that Jack's number is less than the machine's number is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{2023}$

Solution: Jack's number will not be less if the machine chooses a number in the range $[0, 2020)$, which happens with probability $\frac{1010}{1011}$. Otherwise, it is a 50/50 by symmetry, so the probability is

$$\frac{1}{2} \times \left(1 - \frac{1010}{1011}\right) = \frac{1}{2022}$$

for a final answer of $1 + 2022 = \boxed{2023}$.

A5. Let $ABCD$ be a square and P be a point inside it such that the distances from P to sides AB and AD respectively are 2 and 4, while $PC = 6$. If the side length of the square can be expressed in the form $a + \sqrt{b}$ for positive integers a, b, then determine $a + b$.

Proposed by: Andrew Wen

Answer: $\boxed{20}$

Solution:

Let X and Y be the feet from P to AB and AD. Note that if s is the side length, we get $DP^2 = (s-2)^2 + 4^2$ and $BP^2 = (s-4)^2 + 2^2$. By the British Flag Theorem,

$$AP^2 + CP^2 = BP^2 + DP^2 \implies 20 + 36 = 2s^2 - 12s + 40$$

and upon solving and discarding the negative root, we get $s = 3 + \sqrt{17}$ so our answer is $3 + 17 = \boxed{20}$.

Alternatively, we could have let X', Y' be the feet from P to CD, BC, respectively and noted that

$$PX'^2 + PY'^2 = (s-2)^2 + (s-4)^2 = PC^2 = 36$$

which would give the same answer of $s = 3 + \sqrt{17}$ upon solving the quadratic.

A6. Positive integers $a_1, a_2, \dots a_{20}$ sum to 57. Given that M is the minimum possible value of the quantity $a_1!a_2! \dots a_{20}!$, find the number of positive integer divisors of M.

Proposed by: Andrew Wen

Answer: $\boxed{378}$

Solution: We claim that all $a_i \in \{2, 3\}$ in order for the expression to be minimized. This can be done by verifying two things:

- if any a_i is 1, then there must be at least another $a_j \geq 3$, else the sum cannot be 57. Then, we can simply replace (a_i, a_j) with $(2, a_j - 1)$ whilst keeping the sum constant. This decreases the product since

$$a_i!a_j! = 1!a_j! > 2!(a_j - 1)! \iff a_i > \frac{2!}{1!} = 2$$

because $a_j \geq 3$. We repeat this until all $a_i = 1$'s are gone.

- If any $a_i \geq 4$, then there must be another $a_j = 2$, since we know there cannot be $a_j = 1$ and if all $a_j \geq 3$, then the sum far exceeds 57. We can then simply replace (a_i, a_j) with $(a_i - 1, 3)$ whilst keeping the sum constant. This decreases the product since

$$a_i!a_j! = 2!a_i! > 3!(a_i - 1)! \iff a_i > \frac{3!}{2!} = 3$$

since $a_i \geq 4$. We repeat this until all $a_i \geq 4$'s are gone.

The above *smoothing* process does indeed show that the minimum occurs when all a_i's are 2 or 3. Now, if out of the 20 terms, x of them are 2 and y of them are 3, then $x + y = 20, 2x + 3y = 57 \implies x = 3, y = 17$.

Therefore, our minimum product is $M = (2!)^3(3!)^{17} = 2^{20}3^{17}$ with $21 \times 18 = \boxed{378}$ factors.

A7. Bob has 16 balls in a box, where 15 of them are red and one is blue. Bob draws balls out the box three at a time until one of the three is blue. If he ever draws three red marbles, he discards one of them and shuffles the remaining two back into the box. The expected number of draws it takes for Bob to draw the blue ball can be written as a common fraction $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{21}$

Solution: Let the expected number of draws taken for a box with n balls, all red except for one, be $E(n)$. We either draw all reds with probability $\binom{n-1}{3}/\binom{n}{3}$ and then proceed to repeat the same problem for $n - 1$ balls, or end the process with probability $1 - \binom{n-1}{3}/\binom{n}{3}$. Hence, we can derive the following recurrence:

$$\begin{aligned} E(n) &= \left(1 - \frac{\binom{n-1}{3}}{\binom{n}{3}}\right) + \left(\frac{\binom{n-1}{3}}{\binom{n}{3}}\right)(E(n-1) + 1) \\ &= 1 + \frac{\binom{n-1}{3}}{\binom{n}{3}} E(n-1) \\ &= 1 + \left(\frac{n-3}{n}\right) E(n-1). \end{aligned}$$

Since $E(3) = 1$, we can inductively show that $E(n) = \frac{n+1}{4}$; indeed, if $E(k) = \frac{k+1}{4}$, then

$$E(k+1) = 1 + \left(\frac{k-2}{k+1}\right)\left(\frac{k+1}{4}\right) = \frac{k+2}{4}$$

so our answer is $\frac{17}{4} \implies 17 + 4 = \boxed{21}$.

A8. The *Lucas sequence* is defined by these conditions: $L_0 = 2, L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. Determine the remainder when $L_{2019}^2 + L_{2020}^2$ is divided by L_{2023}.

Proposed by: Andrew Wen

Answer: $\boxed{65}$

Solution: We extensively use the sequence definition:

$$\begin{aligned} L_{2020}^2 &= (L_{2022} - L_{2021})^2 \\ &= [L_{2022} - (L_{2023} - L_{2022})]^2 \\ &= (2L_{2022} - L_{2023})^2 \equiv 4L_{2022}^2 \pmod{L_{2023}} \end{aligned}$$

and

$$\begin{aligned} L_{2019}^2 &= (L_{2021} - L_{2020})^2 \\ &= [(L_{2023} - L_{2022}) - (2L_{2022} - L_{2023})]^2 \\ &= (2L_{2023} - 3L_{2022})^2 \equiv 9L_{2022}^2 \pmod{L_{2023}} \end{aligned}$$

Hence $L_{2019}^2 + L_{2020}^2 \equiv 13L_{2022}^2 \pmod{L_{2023}}$. It suffices to find $L_{2022}^2 \pmod{L_{2023}}$. In fact, we can prove that in general, $L_n^2 \equiv 5\cdot(-1)^n \pmod{L_{n+1}}$ with this following extension of Cassini's identity:

Claim: $L_n^2 = L_{n-1}L_{n+1} + 5 \cdot (-1)^n$.

Proof: Note that

$$\begin{aligned} L_n^2 - L_{n-1}L_{n+1} &= L_n(L_{n-2} + L_{n-1}) - L_{n-1}(L_{n-1} + L_n) \\ &= L_{n-2}L_n - L_{n-1}^2 \\ &= -(L_{n-1}^2 - L_{n-2}L_n) \end{aligned}$$

so $L_n^2 - L_{n-1}L_{n+1} = (-1)^{n-1}(L_1^2 - L_0L_2) = 5 \cdot (-1)^n$ as desired. □

So, $L_{2022}^2 \equiv 5 \pmod{L_{2023}}$ so our answer is $13L_{2022}^2 \equiv \boxed{65} \pmod{L_{2023}}$.

A9. Let $ABCD$ be a parallelogram. Point P is selected in its interior such that the distance from P to BC is exactly 6 times the distance from P to AD, and $\angle APB = \angle CPD = 90^\circ$. Given that $AP = 2$ and $CP = 9$, the area of $ABCD$ can be expressed as $m\sqrt{n}$ where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Proposed by: Andy Xu

Answer: $\boxed{46}$

Solution: Note that the height condition tells us that $[BPC] = 6[APD]$. Furthermore, note that $\angle APD + \angle BPC = 360^\circ - 90^\circ - 90^\circ = 180^\circ$. This supplementary angle relation tells us that $\sin(\angle APD) = \sin(\angle BPC)$ so

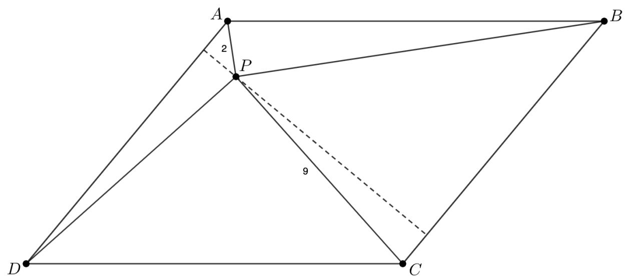
$$\frac{1}{6} = \frac{[APD]}{[BPC]} = \frac{\frac{1}{2} \times AP \times DP \times \sin(\angle APD)}{\frac{1}{2} \times BP \times CP \times \sin(\angle BPC)} = \frac{2DP}{9BP}$$

so it follows that $3BP = 4DP$. Thus, we let $BP = 4x$ and $DP = 3x$ for some x. Now, refer to the diagram below:
By right angles $\angle APB$ and $\angle CPD$, we have

$$AB^2 = CD^2 = 4 + (4x)^2 = 81 + (3x)^2 \implies 7x^2 = 77$$

so $x = \sqrt{11}$ and $BP = 4\sqrt{11}$, $DP = 3\sqrt{11}$. To finish, note that

$$\begin{aligned} [APB] + [CPD] &= \tfrac{1}{2}b \times \text{distance}(P,\, AB) + \tfrac{1}{2}b \times \text{distance}(P,\, BC) \\ &= \tfrac{1}{2}bh \\ &= \tfrac{1}{2}[ABCD] \end{aligned}$$



where $b = AB = CD$ is the horizontal base and h is the vertical height. Finally we can calculate that

$$[APB] + [CPD] = \tfrac{1}{2}(2 \times 4\sqrt{11} + 9 \times 3\sqrt{11}) = \tfrac{35\sqrt{11}}{2}$$

so $[ABCD] = 35\sqrt{11}$, for a final answer of $35 + 11 = \boxed{46}$.

A10. Consider the polynomial $P(x) = x^{35} + \cdots + x + 1$. How many pairs (i, j) of integers are there with $0 \le i < j \le 35$ such that if we flip the signs of the x^i and x^j terms in $P(x)$ to form a new polynomial $Q(x)$, then there exists a nonconstant polynomial $R(x)$ with integer coefficients dividing both $P(x)$ and $Q(x)$?

Proposed by: Andrew Wen

Answer: $\boxed{486}$

Solution: Say we flip x^i and x^j. Then, note that $R(x)$ also divides $P(x) - Q(x) = 2x^i(x^{j-i} + 1)$. Clearly $R(x) \nmid 2x^i$ since $2, x \nmid P(x)$. Thus, $R(x)$ must divide $x^{j-i} + 1$.

- If $j - i$ is odd, then note that $x + 1$ divides both $x^{j-i} + 1$ and $P(x)$ as -1 is a root of both.
- If $j - i$ can be written as $2k$ for an odd k, then note that $x^2 + 1$ divides both $x^{2k} + 1$ (by virtue of $y + 1 \mid y^k + 1$ where $y = x^2$) and $P(x)$ since $x^2 + 1 \mid x^{18} + 1 \mid P(x)$.
- If $j - i$ is a multiple of 4, say $4k$, then we claim $\gcd(x^{4k} + 1, x^{36} - 1) = 1$, which would show $R(x) = 1$, which is impossible. One can check that no 36th root of unity can be a root of $x^{4k} + 1$, as desired.

After exhausting all cases, we see that $R(x)$ exists if and only if $4 \nmid j - i$. We complementary count. Partition $0, 1, \ldots, 35$ into four sets S_1, S_2, S_3, S_4, where S_i consists of all 9 numbers from 0 to 35 that are $i \pmod 4$. Note $4 \mid j - i$ if and only if both are chosen from the same set S_j, of which there are

$$4 \times \binom{9}{2} = 144$$

ways of happening. Therefore, our final answer is $\binom{36}{2} - 144 = 630 - 144 = \boxed{486}$.