MOAA 2022 Gunga Bowl Solutions

Math Open at Andover

October 8th, 2022

G1. A machine outputs the number 2.75 when operated. If it is operated 12 times, then what is the sum of all 12 of the machine outputs?

Proposed by: Andrew Wen

Answer: $\boxed{33}$

Solution: The sum is just $2.75 \cdot 12 = \boxed{33}$.

G2. A car traveling at a constant velocity v takes 30 minutes to travel a distance of d. How long does it take, in minutes, for it travel $10d$ with a constant velocity of $2.5v$?

Proposed by: Andrew Wen

Answer: $\boxed{120}$

Solution: Traveling at a constant velocity of $\frac{5}{2}v$, it must take $\frac{2}{5} \cdot 30 = 12$ minutes to travel a distance of d. Since we need to travel a distance of $10d$, our answer is $10 \cdot 12 = \boxed{120}$.

G3. Andy originally has 3 times as many jelly beans as Andrew. After Andrew steals 15 of Andy's jelly beans, Andy now only has 2 times as many jelly beans as Andrew. Find the number of jelly beans Andy originally had.

Proposed by: Andy Xu

Answer: $\boxed{135}$

Solution: Let x be the number of jelly beans Andrew initially has. Thus, Andy initially has $3x$ jelly beans. After Andrew steals 15 of Andy's jelly beans, Andy now has $3x - 15$ jelly beans and Andrew has $x + 15$ jelly beans. We know that $3x - 15 = 2(x + 15)$ which yields $x = 45$. The answer is then $3x = 3 \cdot 45 = \boxed{135}$.

G4. A coin is weighted so that it is 3 times more likely to come up as heads than tails. How many times more likely is it for the coin to come up heads twice consecutively than tails twice consecutively?

Proposed by: Andrew Wen

Answer: $\boxed{9}$

Solution: A coin can only come up heads or tails, so the probability that it comes up heads is $\frac{3}{4}$ and the probability it comes up tails is $\frac{1}{4}$. The probability we have 2 heads come in a row is $\frac{3}{4} \cdot \frac{3}{4}$ and the probability we have 2 tails come up in a row is $\frac{1}{4} \cdot \frac{1}{4}$. The answer is then $\frac{9}{16} \div \frac{1}{16} = \boxed{9}$.

G5. There are n students in an Areteem class. When 1 student is absent, the students can be evenly divided into groups of 5. When 8 students are absent, the students can evenly be divided into groups of 7. Find the minimum possible value of n.

Proposed by: Andy Xu

Answer: $\boxed{36}$

Solution: Note that $n \equiv 1 \pmod 5$ and $n \equiv 8 \equiv 1 \pmod 7$. This means that $n \equiv 1 \pmod{35}$. The smallest $n > 8$ is thus $\boxed{36}$.

G6. Trapezoid $ABCD$ has $AB \parallel CD$ such that $AB = 5$, $BC = 4$ and $DA = 2$. If there exists a point M on CD such that $AM = AD$ and $BM = BC$, find CD.

Proposed by: Andy Xu

Answer: $\boxed{10}$

Solution: Let the foot of the altitude from A to CD be X and the foot of the altitude from B to CD be Y. Note that $CX = XM$ and $DY = YM$ because $\triangle AMD$ and $\triangle BMC$ are isosceles. However, since $XM + YM = AB = 5$, we know that $CD = 2(XM + YM) = \boxed{10}$.

G7. Angeline has 10 coins (either pennies, nickels, or dimes) in her pocket. She has twice as many nickels as pennies. If she has 62 cents in total, then how many dimes does she have?

Proposed by: Andrew Wen

Answer: $\boxed{4}$

Solution: We set up a system of equations. Let p be the number of pennies, n be the number of nickels, and d be the number of dimes. We know that $p + n + d = 10$, $n = 2p$, and $p + 5n + 10d = 62$. Solving this system yields that $d = \boxed{4}$.

G8. Equilateral triangle ABC has side length 6. There exists point D on side BC such that the area of ABD is twice the area of ACD. There also exists point E on segment AD such that the area of ABE is twice the area of BDE. If $\mathcal{A}$ is the area of triangle ACE, then find $\mathcal{A}^2$.

Proposed by: Andy Xu

Answer: $\boxed{12}$

Solution: Let $[CDE] = x$, where the brackets denote area. It follows that $[BDE] = 2x$, $[ABE] = 4x$, and $[ACE] = 2x$. Thus, $[ACE] = \frac{2x}{x+2x+2x+4x}[ABC] = \frac{2}{9}[ABC] = \frac{2}{9} \cdot 9\sqrt{3} = 2\sqrt{3}$. The answer is then $(2\sqrt{3})^2 = \boxed{12}$.

G9. A number n can be represented in base 6 as $\underline{aba}_6$ and base 15 as $\underline{ba}_{15}$, where a and b are not necessarily distinct digits. Find n.

Proposed by: Andy Xu

Answer: $\boxed{61}$

Solution: The condition is equivalent to $36a + 6b + a = 15b + a$ which simplifies to $b = 4a$. In a base 6 number, we must have $a, b < 6$. Thus, the only possibility is $a = 1$ and $b = 4$. The answer is then $n = 15b + a = \boxed{61}$.

G10. Let $ABCD$ be a square with side length 1. It is folded along a line ℓ that divides the square into two pieces with equal area. The minimum possible area of the resulting shape is $\mathcal{A}$. Find the integer closest to $100\mathcal{A}$.

Proposed by: Andrew Wen

Answer: $\boxed{50}$

Solution: No matter what line we choose, the area of each must be $\frac{1}{2}$. Thus, the answer is $\frac{1}{2} \cdot 100 = \boxed{50}$.

G11. The 10-digit number $\underline{1A2B3C5D6E}$ is a multiple of 99. Find $A+B+C+D+E$.

Proposed by: Andrew Wen

Answer: $\boxed{28}$

Solution: The 10-digit number must be divisible by 9 and 11. To be divisible by 9, we must have the sum of digits be divisible by 9 or equivalently $A+B+C+D+E+17 \equiv 0 \pmod 9 \rightarrow A+B+C+D+E \equiv 1 \pmod 9$. For the number to be divisible by 11, we must have $A+B+C+D+E \equiv 1+2+3+5+6 \equiv 6 \pmod{11}$. Let $A+B+C+D+E = x$ for convenience. We want to find the minimal x so that it is $1 \pmod 9$ and $6 \pmod{11}$. Listing out the numbers that are $6 \pmod{11}$, we see that the minimal $x = 28$. The next biggest x will be $x + 99 = 127$, but $x = A+B+C+D+E <= 5 \cdot 9 = 45$, so the answer is $\boxed{28}$.

G12. Let A, B, C, D be four points satisfying $AB = 10$ and $AC = BC = AD = BD = CD = 6$. If $\mathcal{V}$ is the volume of tetrahedron $ABCD$, then find $\mathcal{V}^2$

Proposed by: Andrew Wen

Answer: $\boxed{200}$

Solution: Let O be the cicumcenter of $\triangle ABD$. The key idea is that O is the projection of C onto $\triangle ABD$. Since $CA = CB = CD$, it follows that A, B, and D lie on a sphere centered at C. In particular, the circumcircle of $\triangle ABD$ is a circular cross section of the sphere. The projection of C onto this cross section will thus be O. The circumradius of $\triangle ABD$ is

$$\frac{AB \cdot BD \cdot DA}{4[ABC]} = \frac{360}{20\sqrt{11}} = \frac{18}{\sqrt{11}}.$$

Since $\triangle AOC$ is a right triangle, we know from Pythagorean Theorem that $CO = \sqrt{AC^2 - AO^2} = \frac{6\sqrt{2}}{\sqrt{11}}$. Therefore, the volume of the tetrahedron is

$$\frac{[ABD] \cdot CO}{3} = \frac{5\sqrt{11} \cdot \frac{6\sqrt{2}}{\sqrt{11}}}{3} = 10\sqrt{2}.$$

The answer is then $(10\sqrt{2})^2 = \boxed{200}$.

G13. Nate the giant is running a 5000 meter long race. His first step is 4 meters, his next step is 6 meters, and in general, each step is 2 meters longer than the previous one. Given that his nth step will get him across the finish line, find n.

Proposed by: Andrew Wen

Answer: $\boxed{70}$

Solution: Note that his nth step will be $4 + 2(n-1) = 2n + 2$ meters long since the first term of the arithmetic sequence is 4 and the common difference is 2. Thus, the total number of meters Nate has traversed after n steps will be

$$\begin{aligned} 4 + 6 + \cdots + 2n + 2 &= 2(2 + 3 + \cdots + n + 1) \\ &= 2\left(\frac{(n+1)(n+2)}{2} - 1\right) \\ &= n^2 + 3n \end{aligned}$$

after simplification. We desire the smallest n such that $n^2 + 3n \geq 5000$, which is $n = \boxed{70}$.

G14. In square $ABCD$ with side length 2, there exists a point E such that $DA = DE$. Let line BE intersect side AD at F such that $BE = EF$. The area of ABE can be expressed in the form $a - \sqrt{b}$ where a is a positive integer and b is a square-free integer. Find $a + b$.

Proposed by: Andy Xu

Answer: $\boxed{5}$

Solution: Since $\triangle ABF$ is a right triangle with right angle at A, it follows that $AE = BE = EF$ because E is the midpoint of BF. This means that $\triangle ABE$ is isosceles, so $\angle BAE = \angle ABE$. This implies that $\angle DAE = \angle CBE$. The additional conditions $DA = CB$ and $AE = BE$ prove that $\triangle DAE \cong \triangle CBE$ by SAS congruence. Since $DE = CE = DC = 2$, we know $\triangle DEC$ is equilateral. The height from E to DC is then $\sqrt{3}$ so the height from E to AB in $\triangle ABE$ is thus $2 - \sqrt{3}$. It follows that the area is $\frac{2\cdot(2-\sqrt{3})}{2} = 2 - \sqrt{3}$ so $a + b = \boxed{5}$.

G15. Patrick the Beetle is located at 1 on the number line. He then makes an infinite sequence of moves where each move is either moving 1, 2, or 3 units to the right. The probability that he does reach 6 at some point in his sequence of moves is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{364}$

Solution: Define p_n to be the probability Patrick reaches 6 at some point if he is currently at n on the number line. We seek p_1. Note that we have the recurrence

$$p_n = \frac{1}{3}p_{n+1} + \frac{1}{3}p_{n+2} + \frac{1}{3}p_{n+3}$$

where $p_6 = 1$ and $p_n = 0$ for $n > 6$. Working backwards from $p_6 = 1$ we can generate

$$p_5 = \frac{1}{3}$$
$$p_4 = \frac{4}{9}$$
$$p_3 = \frac{16}{27}$$
$$p_2 = \frac{37}{81}$$
$$p_1 = \frac{121}{243}$$

giving us an answer of $121 + 243 = \boxed{364}$.

G16. Find the smallest positive integer c greater than 1 for which there do not exist integers $0 \leq x, y \leq 9$ that satisfy $2x + 3y = c$.

Proposed by: Andrew Wen

Answer: $\boxed{44}$

Solution: Note that $x = \frac{c-3y}{2}$. Thus,

$$0 \leq \frac{c-3y}{2} \leq 9$$

$$3y \leq c \leq 3y + 18.$$

We will perform separate cases depending on the parity of c. If c is even, then y must be even so that x is an integer. Thus, $0 \leq y \leq 8$ which means all even $0 \leq c \leq 42$ are attainable. Similarly, if c is odd then y must be odd. Thus, $1 \leq y \leq 9$ so all odd $3 \leq c \leq 45$ are attainable. Combining the inequalities in the two cases implies that the smallest c that cannot be attained is $\boxed{44}$.

G17. Jaeyong is on the point $(0,0)$ on the coordinate plane. If Jaeyong is on point (x,y), he can either walk to $(x+2,y)$, $(x+1,y+1)$, or $(x,y+2)$. Call a walk to $(x+1,y+1)$ an *Brilliant* walk. If Jaeyong cannot have two *Brilliant* walks in a row, how many ways can he walk to the point $(10,10)$?

Proposed by: Andy Xu

Answer: $\boxed{3472}$

Solution: Let a be the number of walks with $(x+2,y)$, b be the number of walks with $(x+1,y+1)$ and c be the number of walks with $(x,y+2)$. It follows that

$$2a + b = 10$$

$$b + 2c = 10$$

which implies $a = c$ and $b = 10 - 2a$. We will perform cases on a. If $a = 5$, then $b = 0$ and the number of ways to arrange 5 a's and 5 c's is $\binom{10}{5} = 252$. If $a = 4$, then $b = 2$. We want the number of ways to arrange 4 a's, 2 b's and 4 c's without having 2 b's in a row. Fix the 2 b's and let x, y, and z be the number of letters to the left of the first b, in between the b's and to the right of the second b respectively. It follows that $x+y+z = 8$ where we require $y \geq 1$. Let $y' = y-1$ so $x+y'+z = 7$ where x, y' and z are nonnegative integers. By Stars and Bars, the number of ways that we can choose x, y' and z is $\binom{7+3-1}{2} = \binom{9}{2}$. The number of ways to rearrange 4 a's and 4 c's once we have fixed the x, y and z is $\binom{8}{4}$ so the number of walks for the case $a = 4$ is $\binom{9}{2} \cdot \binom{8}{4} = 2520$. If $a = 3$, then $b = 4$ and we can use a similar approach to find that the number of walks of this case is $\binom{7}{4} \cdot \binom{6}{3} = 700$. Note that $a < 3$ always result in an arrangement where 2 b's appear in a row, so combining the 3 cases yield an answer of $252 + 2520 + 700 = \boxed{3472}$.

G18. Let $ABCD$ be a square with side length 1. It is folded along a line ℓ that divides the square into two pieces with equal area. The maximum possible area of the resulting shape is $\mathcal{B}$. Find the integer closest to $100\mathcal{B}$.

Proposed by: Andrew Wen

Answer: $\boxed{59}$

Solution: The key idea is that the area of resulting shape is half of the union of 2 squares of unit length sharing a center. This is clear after reflecting the resulting shape over line ℓ. It now suffices to find the maximum area of the union of squares. Fix one of the squares. Observe that the second overlapping square forms 8 congruent right triangles where we just need to maximize the area of each right triangle. Let the right triangle have legs x and y. Thus, we have $x + y + \sqrt{x^2 + y^2} = 1$ from which we solve for y in terms of x to get

$$y = \frac{2x - 1}{2x - 2}.$$

We then have

$$2xy = \frac{2x^2 - x}{x - 1} = 2x + \frac{1}{x - 1} + 1.$$

Let $z = 1 - x$. Substituting, we have

$$2x + \frac{1}{x-1} + 1 = 3 - \left(2z + \frac{1}{z}\right) \leq 3 - 2\sqrt{2}$$

by AM-GM. The area of the union of squares is thus $1 + \frac{1}{2}xy \cdot 4 = 1 + (3 - 2\sqrt{2}) = 4 - 2\sqrt{2}$. The area of region $\mathcal{B}$ is half of the area of the union which is $2 - \sqrt{2}$. Since $100 \cdot (2 - \sqrt{2}) \approx 58.6$ our answer is $\boxed{59}$.

Note 1: Equality occurs when the overlapping square is a $45°$ degree rotation of the first square.

Note 2: One can also solve this geometrically, where the key observation is that each right triangle has a fixed excircle and thus a fixed perimeter. It suffices to maximize the inradius, which is when the incircle is tangent to the excircle. The result follows.

G19. How many ordered triples (x, y, z) with $1 \leq x, y, z \leq 50$ are there such that both $x + y + z$ and $xy + yz + zx$ are divisible by 6?

Proposed by: Andrew Wen

Answer: $\boxed{1753}$

Solution: We will prove that $x \equiv y \equiv z \equiv 0 \pmod 6$, $x \equiv y \equiv z \equiv 2 \pmod 6$, or $x \equiv y \equiv z \equiv 4 \pmod 6$. Let $x + y + z = 6a$ for some positive integer a. Then, $z = 6a - x - y$ and we may substitute it into $6 \mid xy + yz + zx$ to get

$$6 \mid xy + y(6a - x - y) + x(6a - x - y)$$

which is equivalent to

$$6 \mid x^2 + xy + y^2.$$

We can rewrite this as

$$24 \mid (x - y)^2 + 3(x + y)^2$$

which implies $3 \mid x - y$. We will prove $x - y$ cannot be $3 \pmod 6$. Assume the contrary, and consider $6 \mid (x - y)^2 + 3xy$. Since $x - y$ is odd, one of x and y is odd while the other is even. This means that $3xy \equiv 0 \pmod 6$ but $(x - y)^2 \equiv 3 \pmod 6$ so $6 \nmid (x - y)^2 + 3xy$, a contradiction. Therefore, we know $6 \mid x - y$ where x and y are even or else $2 \nmid x^2 + xy + y^2$. If $x \equiv y \equiv 0 \pmod 6$, then $z \equiv 0 \pmod 6$ because $6 \mid x + y + z$. Similarly we find that $x \equiv y \equiv z \equiv 2 \pmod 6$ or $x \equiv y \equiv z \equiv 4 \pmod 6$ giving an answer of $8^3 + 9^3 + 8^3 = \boxed{1753}$.

G20. Triangle ABC has orthocenter H and circumcenter O. If D is the foot of the perpendicular from A to BC, then $AH = 8$ and $HD = 3$. If $\angle AOH = 90°$, find BC^2.

Proposed by: Andy Xu

Answer: $\boxed{160}$

Solution: Define M to be the midpoint of BC. Let H' be the reflection of H about D and let A' be the reflection of H across M. It is well known that H' lies on the circumcircle of $\triangle ABC$ and A' is the antipode of A. Since $HO \perp AA'$ and $AO = OA'$ it follows that $AH = HA' = 8$. Therefore, $HM = 4$ and $DM = \sqrt{HM^2 - HD^2} = \sqrt{4^2 - 3^2} = \sqrt{7}$. Let $BM = x$. This means $BD = x - \sqrt{7}$ and $DC = x + \sqrt{7}$. Power of a Point yields

$$AD \cdot DH' = BD \cdot DC$$

or equivalently

$$11 \cdot 3 = (x - \sqrt{7})(x + \sqrt{7})$$

which implies $x^2 = 40$. Our answer is $BC^2 = (2x)^2 = 4x^2 = \boxed{160}$.

G21. Nate flips a fair coin until he gets two heads in a row, immediately followed by a tails. The probability that he flips the coin exactly 12 times is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{4239}$

Solution: Note that Nate must flip the coin 9 times such that there is no substring of HHT and then flip HHT for the last 3 flips. Let h_n be the number of strings of length n starting with H so that HHT does not appear and let t_n be the number of strings of length n starting with T so that HHT does not appear. We seek $h_9 + t_9$. Assuming we start with H, if the next letter is T then we have t_{n-1} ways to fill the rest of the string. If the next letter is H, then the string must be all H so there is only 1 way. Thus, we have

$$h_n = t_{n-1} + 1.$$

Assuming we start with T, if the next letter is H then we have h_{n-1} ways to fill the rest of the string. If the next letter is T then we have t_{n-1} ways. Thus, we have

$$t_n = h_{n-1} + t_{n-1}.$$

Substituting our first recurrence into the second recurrence, we arrive at

$$t_n = t_{n-1} + t_{n-2} + 1$$

where $t_1 = 1$ and $t_2 = 2$. Using this recursion we can generate $t_8 = 54$ and $t_9 = 88$. Since $h_9 = t_8 + 1 = 55$, we have $h_9 + t_9 = 55 + 88 = 143$. The probability is then $\frac{143}{4096}$ giving an answer of $143 + 4096 = \boxed{4239}$.

G22. Let f be a function defined by $f(1) = 1$ and

$$f(n) = \tfrac{1}{p} f(\tfrac{n}{p}) f(p) + 2p - 2,$$

where p is the least prime dividing n, for all integers $n \geq 2$. Find $f(2022)$.

Proposed by: Andrew Wen

Answer: $\boxed{2706}$

Solution: Primes are very important to the structure of this function. Indeed, note that

$$f(p) = \frac{1}{p} f(1) f(p) + 2p - 2 \implies f(p)\left(1 - \frac{1}{p}\right) = 2(p-1) \implies f(p) = 2p$$

for any prime p. So more generally, for all n it follows that $f(n) = 2f\left(\frac{n}{p}\right) + 2(p-1)$ where p is the minimal prime dividing it. Thus,

$$f(2022) = 2f(1011) + 2 = 2(2f(337) + 4) + 2 = 2(2 \cdot 337 \cdot 2 + 4) + 2 = \boxed{2706}.$$

G23. Jessica has 15 balls numbered 1 through 15. With her left hand, she scoops up 2 of the balls. With her right hand, she scoops up 2 of the remaining balls. The probability that the sum of the balls in her left hand is equal to the sum of the balls in her right hand can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Andy Xu

Answer: $\boxed{614}$

Solution: Let Jessica choose 4 balls such that she has balls numbered a and d in her left hand and balls numbered b and c in her right hand. Note that we must have $a < b < c < d$ so that $a + d = b + c$. We can rewrite this relation as $a - b = c - d$. Let a_1, a_2, a_3, a_4, and a_5 be the number of balls before ball a, in between balls a and b, in between balls b and c, in between balls c and d, and after d respectively. It follows that $a_1 + a_2 + a_3 + a_4 + a_5 = 11$ and that $a_2 = a_4$ since $a - b = a_2 + 1 = c - d = a_4 + 1$. Therefore, we have

$$a_1 + a_3 + a_5 = 11 - 2a_2.$$

If we fix a_2, the number of ways to choose a_1, a_3, and a_5 is $\binom{11-2a_2+3-1}{2} = \binom{13-2a_2}{2}$ by a Stars and Bars argument. Since $0 \le a_2 \le 5$, the total number of ways that satisfy the condition is

$$\binom{13}{2} + \binom{11}{2} + \binom{9}{2} + \binom{7}{2} + \binom{5}{2} + \binom{3}{2} = 203.$$

The total number of ways Jessica could choose the 4 balls is $\frac{1}{2}\binom{15}{2}\binom{13}{2} = 4095$ so the probability is $\frac{203}{4095} = \frac{29}{585}$ giving an answer of $29 + 585 = \boxed{614}$.

G24. Let $ABCD$ be a cyclic quadrilateral such that its diagonal $BD = 17$ is the diameter of its circumcircle. Given $AB = 8$, $BC = CD$, and that a line ℓ through A intersects the incircle of ABD at two points P and Q, the maximum area of CPQ can be expressed as a fraction $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{73}$

Solution: Draw the radii IP and IQ. Note that $\triangle IPQ$ and $\triangle CPQ$ share the same base PQ. Let X be the projection from I onto line PQ and let Y be the projection from C onto line PQ. Observe that $\triangle AIX \sim \triangle ACY$ since $IX \parallel CY$. It follows that

$$\frac{[CPQ]}{[IPQ]} = \frac{CY}{IX} = \frac{AC}{AI}.$$

By the Incenter-Excenter Lemma, we know that $IC = BC = CD = \frac{17}{\sqrt{2}}$. Furthermore, Ptolemy's Theorem yields

$$8 \cdot \frac{17}{\sqrt{2}} + 15 \cdot \frac{17}{\sqrt{2}} = 17AC \implies AC = \frac{23}{\sqrt{2}}.$$

Therefore, $AI = AC - IC = \frac{23}{\sqrt{2}} - \frac{17}{\sqrt{2}} = \frac{6}{\sqrt{2}}$. Substituting, we have

$$\frac{[CPQ]}{[IPQ]} = \frac{23}{6} \implies [CPQ] = \frac{23}{6}[IPQ]$$

so it suffices to maximize $[IPQ]$. Note that $IP = IQ = \frac{8+15-17}{2} = 3$ so the maximum of $[IPQ]$ is $\frac{9}{2}$ when $\angle PIQ = 90^\circ$. It follows that the maximum of $[CPQ] = \frac{23}{6} \cdot \frac{9}{2} = \frac{69}{4}$ yielding an answer of $69 + 4 = \boxed{73}$.