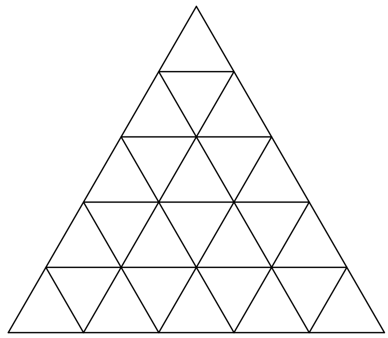
MOAA 2022 Team Round Solutions

Math Open at Andover

October 8th, 2022

T1. Consider the 5 by 5 equilateral triangular grid as shown:



How many equilateral triangles are there with sides along the gridlines?

Proposed by: Andy Xu

Answer: $\boxed{48}$

Solution: Casework count:

- 1 by 1: there are 15 upright, 10 upside down for a total of 25.
- 2 by 2: there are 10 upright, 3 upside down for a total of 13.
- 3 by 3: there are 6 upright, none upside down for a total of 6.
- 4 by 4: there are 3 upright, none upside down for a total of 3.
- 5 by 5: there is 1 upright, none upside down for a total of 1.

Summing them all yields $25 + 13 + 6 + 3 + 1 = \boxed{48}$.

T2. Jeff draws two intersecting segments $AB = 10$ and $CD = 7$ on a plane. Determine the maximum possible area of quadrilateral $ACBD$.

Proposed by: Andrew Wen

Answer: $\boxed{35}$

Solution: Note

$$\begin{aligned}[ACBD] &= [CAD] + [CBD] \\ &= \frac{1}{2} \cdot [7\delta(A, CD)] + \frac{1}{2} \cdot [7\delta(B, CD)] \\ &= \frac{7}{2}[\delta(A, CD) + \delta(B, CD)]\end{aligned}$$

where $\delta(X, \ell)$ denotes distance from point X to line ℓ. The quantity in parentheses is at most the length of AB, achieved when $AB \perp CD$, hence $[ACBD] \le \frac{7}{2} \cdot 10 = \boxed{35}$.

T3. The area of the figure enclosed by the x-axis, y-axis, and line $7x + 8y = 15$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{337}$

Solution: This is really just a right triangle with vertices at the origin and the intercepts of $7x + 8y = 15$. By setting x and y to be 0, one at a time, we find that the x intercept is $\frac{15}{7}$ and the y intercept is $\frac{15}{8}$, so the area is $\frac{1}{2} \cdot \frac{15}{7} \cdot \frac{15}{8} = \frac{225}{112}$ for a final answer of $225 + 112 = \boxed{337}$.

T4. Andy flips three fair coins, and if there are any tails, he then flips all coins that landed tails each one more time. The probability that all coins are now heads can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Andrew Wen

Answer: $\boxed{91}$

Solution: We view each coin independently; it either is heads on its first flip with a chance of $\frac{1}{2}$, or it can redeem itself on the second flip with a chance of $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ since it must have been tails on the first flip (to even get a second flip at all) and heads on the second.

Hence, each coin has a $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ chance of being heads after two iterations, so our final probability is $\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$ for a final answer of $27 + 64 = \boxed{91}$.

T5. Find the smallest positive integer that is equal to the sum of the product of its digits and the sum of its digits.

Proposed by: Andrew Wen

Answer: $\boxed{19}$

Solution: The number is at least two digits because for a single digit number n, the quantity in question is $2n > n$.

For a two digit number $n = 10a + b$, note that we want

$$n = 10a + b = ab + a + b \implies 9a = ab$$

which either gives $a = 0$, impossible, or $b = 9$. There are no restrictions on a, so the minimum answer is $\boxed{19}$. Indeed, check $19 = 1 \cdot 9 + 9 + 1$.

T6. Define a positive integer n to be *almost-cubic* if it becomes a perfect cube upon concatenating the digit 5. For example, 12 is *almost-cubic* because $125 = 5^3$. Find the remainder when the sum of all *almost-cubic* $n < 10^8$ is divided by 1000.

Proposed by: Andy Xu

Answer: $\boxed{950}$

Solution: If x is almost-cubic, then $10x + 5$, which is the value formed upon concatenating a 5 can be written in the form $(10n - 5)^3$ for some positive integer n. Hence, we are really just looking

$$\sum_{i=1}^{N} \frac{(10i - 5)^3 - 5}{10}$$

where N is the greatest value for which $\frac{(10N-5)^3-5}{10} < 10^8 \implies (10N - 5) < 1000 + \epsilon$ so $N = 100$.

Hence, our sum is

$$\sum_{i=1}^{100} \frac{(10i-5)^3-5}{10} = \sum_{i=1}^{100}(100i^3 - 150i^2 + 75i - 13)$$

which upon taking mod 1000, becomes

$$\begin{aligned} -150\left[\frac{100\cdot 101\cdot 201}{6}\right] + 75\left[\frac{100\cdot 101}{2}\right] - 1300 &\equiv -500+750-300 \pmod{1000} \\ &\equiv \boxed{950} \pmod{1000} \end{aligned}$$

as desired.

T7. A point P is chosen uniformly at random in the interior of triangle ABC with side lengths $AB = 5$, $BC = 12$, $CA = 13$. The probability that a circle with radius $\frac{1}{3}$ centered at P does not intersect the perimeter of ABC can be written as $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m+n$.

Proposed by: Andrew Wen

Answer: $\boxed{61}$

Solution: The locus of the centers is the set of all points at least $\frac{1}{3}$ away from the perimeter of ABC. This is a triangle, say $A'B'C'$ (labeled by $AB \parallel A'B'$, etc). Note that A' is equidistant (specifically having distance of $\frac{1}{3}$) from AB and AC, hence AA' is the $\angle A$ bisector. Similarly, BB' and CC' are $\angle B$ and $\angle C$ bisectors, so AA', BB', CC' concur at incenter I. Since $A'B'C'$ and ABC have pairwise parallel sides, there is a homothety (or just dilation) at I shrinking ABC to $A'B'C'$.

The scale factor is simply

$$\frac{\delta(I, A'B')}{\delta(I, AB)} = \frac{r'}{r} = \frac{r-\frac{1}{3}}{r}$$

where I is the incenter of ABC, $\delta(X, \ell)$ is the distance from point X to line ℓ, and r, r' are the inradii of ABC and $A'B'C'$ respectively. Note that by calculating the area of ABC twice, we have $(5+12+13)r = 5\cdot 12 \implies r = 2$ so scale factor is $\frac{5}{6}$.

Therefore, the scale factor between areas is $\left(\frac{5}{6}\right)^2 = \frac{25}{36}$ for an answer of $25+36 = \boxed{61}$.

T8. Freddy the frog is playing a game in a circular pond with six lilypads around its perimeter numbered clockwise from 1 to 6 (so that pad 1 is adjacent to pad 6). He starts at pad 1, and when he is on pad i, he may jump to one of its two adjacent pads, or any pad labeled with j for which $j - i$ is even. How many jump sequences enable Freddy to hop to each pad exactly once?

Proposed by: Andy Xu

Answer: $\boxed{40}$

Solution: Note that from any lilypad, each frog can literally make any jump except to the pad across from it. We are essentially looking for a permutation starting with 1 without the subsequences $(1,4)$, $(2,5)$, and $(3,6)$. The key idea is to complementary count: we will find the number of permutations that have at least one of $(1,4)$, $(2,5)$ and $(3,6)$. Note that by symmetry we can just find the number of permutations starting with any number and then divide by 6 at the end.

We use Principle of Inclusion-Exclusion:

- $(1,4)$ exists: Treat it as one block, where we permute the block and the other 4 numbers. The number of ways to do this is $5!$, but note that we can swap the 1 and 4 in the block so there are $5! \cdot 2 = 240$ orders in this case. By symmetry there are 240 for $(2,5)$ and $(3,6)$.
- $(1,4)$ and $(2,5)$ both exist: Treating $(1,4)$ and $(2,5)$ both as blocks, when we consider swapping, we have $4! \cdot 2 \cdot 2 = 96$ orders. Similarly, if $(1,4)$ and $(3,6)$ exist there are 96 orders and if $(2,5)$ and $(3,6)$ exist there are 96 orders.
- All pairs exist: Treat each pair as a block. The number of orders is thus $3! \cdot 2 \cdot 2 \cdot 2 = 48$

The final computation yields $240 + 240 + 240 - 96 - 96 - 96 + 48 = 480$. The number of permutations that start with a 1 and do not have any of the pairs $(1,4)$, $(2,5)$ and $(3,6)$ is thus $\frac{720-480}{6} = \boxed{40}$.

T9. Davin has two cups A and B, each of which can hold 400 mL, A initially with 200 mL of water and B initially with 300 mL of water. During a *round*, he chooses the cup with more water (randomly picking if they have the same amount), drinks half of the water in the chosen cup, then pours the remaining half into the other cup and refills the chosen cup to back to half full. If Davin goes for 20 rounds, how much water does he drink, to the nearest integer?

Proposed by: Andrew Wen

Answer: $\boxed{3900}$

Solution: Let (A, B) denote the pair consisting of the water volumes (mL) in A and B, not necessarily ordered.

Note that from any given state $(200, n)$ with $200 \le n \le 400$, Davin drinks $\frac{n}{2}$ and the state transforms to $(200 + \frac{n}{2}, 200) \iff (200, 200 + \frac{n}{2})$.

Note n starts as 300, and that $200 + \frac{n}{2} > n$ when $n > 200$. It follows that the non-200 cup's volume is continuously increasing. We can model the state of the cups after rounds as $(200, a_0), (200, a_1), \ldots$ for an increasing sequence a_i, with $a_0 = 300$ as well as $a_{i+1} = 200 + \frac{a_i}{2}$.

Through induction (or really just pattern observation), one can conclude that

$$a_i = \left(\frac{2^i - 1}{2^i}\right) \cdot 400 + \left(\frac{1}{2^i}\right) \cdot a_0 = 400 + \frac{a_0 - 400}{2^i} = 400 - \frac{100}{2^i}.$$

Note that in round i, he drinks $\frac{a_i}{2} = 200 - \frac{50}{2^i}$. Since he goes for 20 rounds, he starts at round 0 and ends at round 19, hence the total volume drank is

$$\sum_{i=0}^{19} \left(200 - \frac{50}{2^i}\right) = 4000 - 50\left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{19}}\right) = 3900 + \frac{50}{2^{19}} \approx \boxed{3900}$$

as desired.

T10. Three integers A, B, C are written on a whiteboard. Every move, Mr. Doba can either subtract 1 from all numbers on the board, or choose two numbers on the board and subtract 1 from both of them whilst leaving the third untouched. For how many ordered triples (A, B, C) with $1 \le A < B < C \le 20$ is it possible for Mr. Doba to turn all three of the numbers on the board to 0?

Proposed by: Yifan Kang

Answer: $\boxed{615}$

Solution: If $A+B \geq C$, we can first subtract $\lfloor \frac{A+B-C}{2} \rfloor$ from A, B , $\lfloor \frac{A+C-B}{2} \rfloor$ from A, C , and $\lfloor \frac{B+C-A}{2} \rfloor$ from B, C . If $A+B+C$ is even, we are done. Otherwise, we subtract 1 from A, B, C and we are done.

If $A+B < C$, notice that every time we subtract 1 from C , the value of $A+B$ at least decreases by 1 . Thus, $A+B$ will gets to 0 before C gets to 0 , and (A, B, C) is invalid.

Denote the answer for $1 \leq A < B < C \leq 2n$ as $f(n)$. Next, we can prove by induction that:

$$f(n) = \sum_{k=0}^{n-1} \binom{2k+1}{2} = \sum_{k=0}^{n-1} (2k^2 + k) = \frac{(n-1)n(2n-1)}{3} + \frac{n(n-1)}{2}$$

To prove this, notice that $f(1) = 0, f(2) = 3, f(3) = 3 + 10 = 13$. It suffices to prove that $f(n+1) - f(n) = n(2n+1)$. This can be done by counting the number of (A, B, C) with $C = 2n+1$ and $C = 2n+2$.

Finally, the answer for this problem is $f(10) = \frac{9 \times 10 \times 19}{3} + \frac{10 \times 9}{2} = \boxed{615}$.

T11. Let a *triplet* be some set of three distinct pairwise parallel lines. 20 *triplets* are drawn on a plane. Find the maximum number of regions these 60 lines can divide the plane into.

Proposed by: Andrew Wen

Answer: $\boxed{1771}$

Solution: Let the answer for n triplets be a_n. Note that $a_0 = 1$. We claim that

$$a_{n+1} = a_n + 3(3n+1)$$

holds. Already given n triplets , we will show that adding any one more line adds at most $3n+1$ new regions. Indeed, say this new line does not pass through any existing intersection for optimality, and it intersects the existing $3n$ lines at $P_1, P_2, \ldots, P_{3n}$ from left to right. Note that these $3n$ points partition the new line into $3n+1$ sections, where each section divides the region containing it in two, as desired.

During each step of the recursion, we add three such lines that cannot intersect each other, so we add $3(3n+1)$ new regions each time, proving the recursion.

Now, we evaluate:

$$\begin{aligned} a_{20} &= a_{19} + 3(58) \\ &= a_{18} + 3(58 + 55) \\ &\vdots \\ &= a_0 + 3(1 + 4 + \ldots + 58) \\ &= 1 + 3(10 \cdot 59) \end{aligned}$$

for a final answer of $\boxed{1771}$.

T12. Triangle ABC has circumcircle ω where B' is the point diametrically opposite B and C' is the point diametrically opposite C. Given $B'C'$ passes through the midpoint of AB, if $AC' = 3$ and $BC = 7$, find AB'^2.

Proposed by: Andy Xu

Answer: $\boxed{30}$

Solution: First note that $B'C' \parallel BC$ so $B'C'$ also goes through the midpoint of AC. Let the midpoint of AB be M and the midpoint of AC be N. The key is that the orthocenter is reflection of B' about N and C' about M. Denote the orthocenter as H. Since $AM = MB$ and $C'M = MH$, we know that $AC'BH$ is a parallelogram. This means that $BH = AC' = 3$. Let AH intersect BC at D. Since $C'HDB$ is a rectangle, we know that $C'D = BH = 3$. Now, let the nine point circle of $\triangle ABC$ passing through M, N and D be denoted as ω. For the computation, let $C'M = a$ and $B'N = b$. Since ω passes through the midpoint of BH, by Power of a Point with respect to ω we have

$$a(2a+b) = \frac{C'D}{2} \cdot C'D = \frac{3}{2} \cdot 3 = \frac{9}{2}.$$

Additionally, we know that

$$B'C' = 2a + 2b = 7.$$

Solving the system yields that $a = 1$ and $b = \frac{5}{2}$. By Pythagorean Theorem we know $AH = \sqrt{5}$ and thus $AB'^2 = AH^2 + B'H^2 = (\sqrt{5})^2 + 5^2 = \boxed{30}$.

T13. Determine the number of distinct positive real solutions to

$$\lfloor x \rfloor^{\{x\}} = \frac{1}{2022}x^2.$$

Note: $\lfloor x \rfloor$ is known as the *floor* function, which returns the greatest integer less than or equal to x. Furthermore, $\{x\}$ is defined as $x - \lfloor x \rfloor$.

Proposed by: Andrew Wen

Answer: $\boxed{1975}$

Solution: We figure out the criterion for there being a solution in $[n, n+1)$. Note that a graphical representation of $\lfloor x \rfloor^{\{x\}}$ restricted to $[n, n+1)$ would simply just be a steep, concave-up curve from $(n, 1)$ infinitesimally nearing $(n+1, n)$ but never actually hitting it.

In order for $f(x) = \frac{1}{2022}x^2$ to have a singular solution in $[n, n+1)$, it must intersect the induced curve from $(n, 1)$ to $(n+1, n)$ once, which only happens if $f(n) \geq 1$ and $f(n+1) < n$.

$$f(n) \geq 1 \iff \frac{n^2}{2022} \geq 1 \iff n \geq \sqrt{2022} \implies n \geq 45$$

and

$$f(n+1) < n \iff (n+1)^2 < 2022n \iff n^2 - 2020n + 1 < 0 \implies n \leq 2019.$$

It is clearly impossible to have two solutions in a singular interval, since as n gets large, the induced curve from $(n, 1)$ to $(n+1, n)$ is much more steep than the parabola, so there is a singular solution in each interval $[45, 46), [46, 47), \ldots, [2019, 2020)$ for a total of $\boxed{1975}$ solutions.

T14. Find the greatest prime number p for which there exists a prime number q such that p divides $4^q + 1$ and q divides $4^p + 1$.

Proposed by: Andrew Wen

Answer: $\boxed{41}$

Solution: A quick check shows that p, q must be odd. Without loss of generality, let $p \geq q$. By Fermat's Little Theorem, we have

$$2^{(p+1)q} + 1 \equiv 4^q + 1 \equiv 0 \pmod p$$

$$2^{(q+1)p} + 1 \equiv 4^p + 1 \equiv 0 \pmod q$$

hence $4^q \equiv -1 \pmod p \implies 16^q \equiv 1 \pmod p$ and $4^p \equiv -1 \pmod q \implies 16^p \equiv 1 \pmod q$. Therefore, if we let e_1 be the order of $16 \pmod p$ and e_2 be the order of $16 \pmod q$, we have

$$e_1 |\, \gcd(q, p-1)$$

$$e_2 |\, \gcd(p, q-1).$$

Note that $p \geq q \implies p > q - 1$. This implies that $e_2 | 1$; if not, then $p | q - 1$ which is clearly not possible. Hence, $e_2 = 1$ and the order of $16 \pmod q$ is 1 and $q | 15 \implies q = 3, 5$.

Suppose $q = 3$. Then, $3 | 16^p + 1$, which is clearly not possible since $16^p + 1 \equiv 2 \pmod 3$.

Suppose $q = 5$. Then,

$$32^{p+1} \equiv -1 \pmod p$$

but since $p \geq 5$ we may use Fermat's Little Theorem and get $32^2 \equiv -1 \pmod p$ Therefore, $p | 1025 \implies p = 5, 41$. A quick check shows that both work.

Therefore, our solutions are $(p, q) = (5, 5), (5, 41), (41, 5)$ and the greatest prime p is $\boxed{41}$.

T15. Let I_B, I_C be the B, C-excenters of triangle ABC, respectively. Let O be the circumcenter of ABC. If BI_B is perpendicular to AO, $AI_C = 3$ and $AC = 4\sqrt{2}$, then AB^2 can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Note: In triangle $\triangle ABC$, the *A-excenter* is the intersection of the exterior angle bisectors of $\angle ABC$ and $\angle ACB$. The B-excenter and C-excenter are defined similarly.

Proposed by: Andy Xu

Answer: $\boxed{697}$

Solution: Let I be the incenter of $\triangle ABC$ and let the CI_C intersect the circumcircle of $\triangle ABC$ at X. We will first prove $\angle B = 2\angle C$. Note that by the Incenter-Excenter Lemma we know that I_C is the reflection of I over X and $AX = BX = IX$. However, $XI = XA = XB = XI_C$ implies $\angle I_C AI = 90°$ and $\angle I_B BI_C = 90°$. Thus, $AO \perp BI_B$ implies that $AO \parallel I_C B$ which means $\angle I_B AO = \angle I_B I_C B$. Note that $\angle I_B I_C B = 180 - \angle AIB = 180 - \left(90 + \frac{\angle C}{2}\right) = 90 - \frac{\angle C}{2}$ since $AIBI_C$ is cyclic. Additionally, $\angle I_B AO = \angle CAO + \angle I_B AC = 90 - \angle B + \angle I_B IC$ since $AICI_B$ is cylic. Since $\angle I_B IC = 180 - \angle BIC = 180 - \left(90 + \frac{\angle A}{2}\right) = 90 - \frac{\angle A}{2}$, we know $\angle I_B AO = 90 - \angle B + 90 - \frac{\angle A}{2} = 180 - \angle B - \frac{\angle A}{2}$. Therefore,

$$90 - \frac{\angle C}{2} = 180 - \angle B - \frac{\angle A}{2} = 180 - \angle B - \frac{180 - \angle B - \angle C}{2} = 90 - \frac{\angle B}{2} + \frac{\angle C}{2}$$

which implies $\angle B = 2\angle C$ as desired.

Let line $I_B I_C$ to meet line BC at P. Note that $\angle AI_C C = \angle ABI = \frac{\angle B}{2} = \angle C$ so $\angle I_C PC = \angle AI_C C - \angle I_C CP = \angle C - \frac{\angle C}{2} = \frac{\angle C}{2}$. Therefore, $\triangle I_C PC$ is isosceles. Additionally, observe that $\triangle AI_C C \sim \triangle ACP$ because $\angle AI_C C = \angle ACP = \angle C$ and $\angle I_C AC = \angle CAP$. Thus, $\frac{AI_C}{AC} = \frac{AC}{AP}$ which yields $AP = \frac{32}{3}$ and $I_C P = \frac{32}{3} - 3 = \frac{23}{3} = I_C C$. The similarity also implies $\frac{AI_C}{I_C C} = \frac{AC}{PC}$ which yields $PC = \frac{92\sqrt{2}}{9}$.

Now, let the line tangent to the circumcircle of $\triangle ABC$ at A intersect line BC at Q. It follows that $\angle QAB = \angle C$ and $\angle AQB = \angle ABC - \angle QAB = 2\angle C - \angle C = \angle C$. This means that $\triangle AQB$ is isosceles with $AB = QB$ and $\triangle AQC$ is also isosceles with $AC = AQ$. Additionally, $\angle PAQ = \angle AQC - \angle APQ = \angle C - \frac{\angle C}{2} = \frac{\angle C}{2}$ so $\triangle APQ$ is isosceles. Thus,

$$QC = PC - PQ = PC - AQ = PC - AC = \frac{92\sqrt{2}}{9} - 4\sqrt{2} = \frac{56\sqrt{2}}{9}.$$

Power of a Point yields $AQ^2 = QB \cdot QC$ which is equivalently

$$AC^2 = AB \cdot QC.$$

Therefore,

$$AB = \frac{AC^2}{QC} = \frac{32}{\frac{56\sqrt{2}}{9}} = \frac{18\sqrt{2}}{7}$$

so $AB^2 = \frac{648}{49}$ giving an answer of $648 + 49 = \boxed{697}$.