MOAA 2023 Accuracy Round Solutions

Math Open at Andover

October 7, 2023

A1. Compute

$$\left(20+\frac{1}{23}\right)\cdot\left(23+\frac{1}{20}\right)-\left(20-\frac{1}{23}\right)\cdot\left(23-\frac{1}{20}\right)$$

Proposed by: Andy Xu

Answer: $\boxed{4}$

Solution: For simplicity let $x = 20$ and $y = 23$. The above is equivalent to

$$\left(x+\frac{1}{y}\right)\cdot\left(y+\frac{1}{x}\right)-\left(x-\frac{1}{y}\right)\cdot\left(y-\frac{1}{x}\right)=xy+\frac{1}{xy}+2-\left(xy+\frac{1}{xy}-2\right)=\boxed{4}$$

A2. Let $ABCD$ be a square. Let M be the midpoint of BC and N be the point on AB such that $2AN = BN$. If the area of $\triangle DMN$ is 15, find the area of square $ABCD$.

Proposed by: Harry Kim

Answer: $\boxed{36}$

Solution: Let s be the side length square $ABCD$. Then, $AN = \frac{s}{3}$, $BN = \frac{2s}{3}$, and $BM = CM = \frac{s}{2}$. From this, we get $[AND] = \frac{s^2}{6}$, $[BMN] = \frac{s^2}{6}$, and $[CDM] = \frac{s^2}{4}$ where $[XYZ]$ represents the area of ΔXYZ. Summing up these areas yields

$$\frac{s^2}{6}+\frac{s^2}{6}+\frac{s^2}{4}+15=\frac{7s^2}{12}+15=s^2$$

so $\frac{5s^2}{12} = 15 \Rightarrow s^2 = 36$. Since we want the area of $ABCD$, the answer is $\boxed{36}$.

A3. Ms. Raina's math class has 6 students, including the troublemakers Andy and Harry. For a group project, Ms. Raina randomly divides the students into three groups containing 1, 2, and 3 people. The probability that Andy and Harry unfortunately end up in the same group can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Andy Xu

Answer: $\boxed{19}$

Solution: Note that it only matters where Andy and Harry get placed. The probability that they both get assigned to the group with 2 people is $\frac{\binom{2}{2}}{\binom{6}{2}} = \frac{1}{15}$ and the probability that they both get assigned to the group with 3 people is $\frac{\binom{3}{2}}{\binom{6}{2}} = \frac{3}{15}$ so the probability is $\frac{1}{15} + \frac{3}{15} = \frac{4}{15}$ which means the answer is $4 + 15 = \boxed{19}$.

A4. A two-digit number $\overline{ab}$ is *self-loving* if a and b are relatively prime positive integers and $\overline{ab}$ is divisible by $a+b$. How many *self-loving* numbers are there?

Proposed by: Anthony Yang and Andy Xu

Answer: $\boxed{8}$

Solution: Note that $\overline{ab} = 10a + b$ so our condition is

$$a + b \mid 10a + b$$

where the notation $x \mid y$ means that x divides y. This relation can be rewritten as

$$a + b \mid 10a + b - (a + b) = 9a$$

but since $\gcd(a, b) = 1$ it follows that $\gcd(a + b, a) = \gcd(b, a) = 1$ so $a + b \mid 9$. If $a + b = 9$ then we have 6 possible ordered pairs of (a, b) since we exclude $(3, 6)$ and $(6, 3)$. If $a + b = 3$ then we just have 2 possible pairs $(1, 2)$ or $(2, 1)$. If $a + b = 1$ then there are no possible pairs. The answer is then $6 + 2 = \boxed{8}$.

A5. Let k be a constant such that exactly three real values of x satisfy

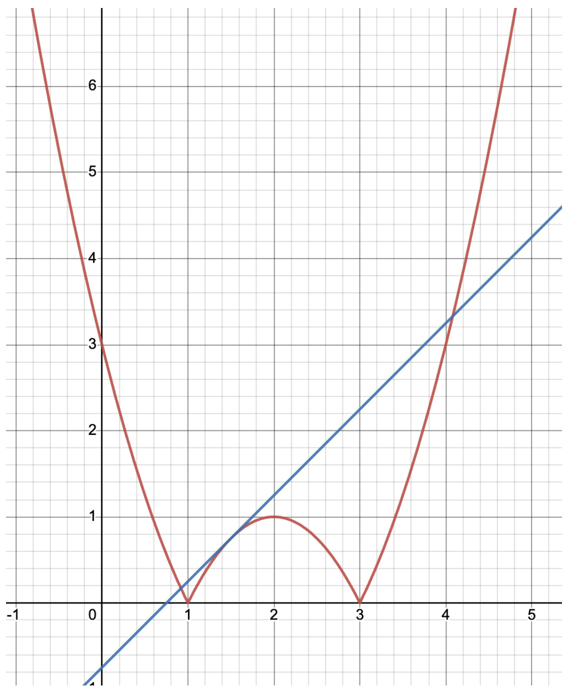
$$x - |x^2 - 4x + 3| = k$$

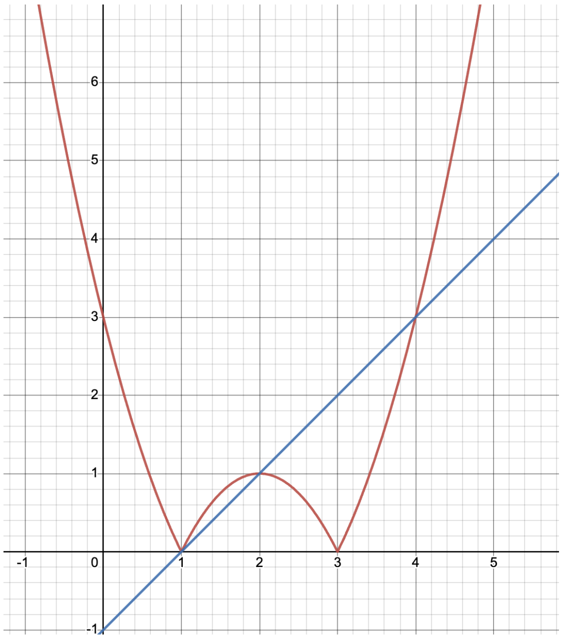
The sum of all possible values of k can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m + n$.

Proposed by: Andy Xu

Answer: $\boxed{11}$

Solution: There are many ways to solve this problem algebraically or graphically. We present the following graphing solution. Rearrange the relation to $|x^2 - 4x + 3| = x - k$. After graphing the graph of $y = |x^2 - 4x + 3|$, it's clear that there are only two cases in which the line $y = x - k$ intersects the graph of $y = |x^2 - 4x + 3|$ as shown below.





Now, it suffices to find the values of k that correspond to these lines. In the first case, it's clear that the line $y = x - k$ is tangent to the graph of $y = -x^2 + 4x - 3$. This implies that the equation $-x^2 + 4x - 3 = x - k$ has only one solution for x. So, the quadratic $x^2 - 3x + (3 - k) = 0$ has one solution. This means the discriminant $9 - 4(3 - k) = 0$ yielding $k = \frac{3}{4}$.

In the second case, it's easy to see that $k = 1$ since the line $y = x - k$ passes through the point $(1, 0)$. Thus, the sum of all possible values of k is $\frac{3}{4} + 1 = \frac{7}{4}$ so the answer is $7 + 4 = \boxed{11}$.

A6. Let b be a positive integer such that 2032 has 3 digits when expressed in base b. Define the function $S_k(n)$ as the sum of the digits of the base k representation of n. Given that $S_b(2032) + S_{b^2}(2032) = 14$, find b.

Proposed by: Anthony Yang

Answer: $\boxed{26}$

Solution: Let $2032_{10} = \overline{pqr}_b$, or $2032 = p \cdot b^2 + q \cdot b + r$. Note that

$$p \cdot b^2 + q \cdot b + r = p \cdot (b^2)^1 + (q \cdot b + r) \cdot (b^2)^0.$$

Since $q \cdot b + r < b^2$, the digits of 2032 when expressed in base b^2 are p and $q \cdot b + r$. Thus, we have

$$S_b(2032) + S_{b^2}(2032) = (p + q + r) + (p + (q \cdot b + r)) = 2p + q \cdot (b + 1) + 2r = 14.$$

Now, notice that $b^2 \leq 2032 < b^3 \Rightarrow 13 \leq b \leq 44$. If $q > 0$, then

$$q \cdot (b + 1) \geq 14 \Rightarrow 2p + q \cdot (b + 1) + 2r > 14,$$

so we have $q = 0$. We now have the following equations:

$$p + r = 7$$
$$p \cdot b^2 + r = 2032$$

Subtracting the first equation from the second yields $p \cdot (b^2 - 1) = 2025$. Since $b \geq 13$, the only possible values of p are 1, 3, 5, and 9. Testing these values gives $(p, b) = (3, 26)$ as the only solution, so $b = \boxed{26}$.

A7. Pentagon $ANDD'Y$ has $AN \parallel DY$ and $AY \parallel D'N$ with $AN = D'Y$ and $AY = DN$. If the area of $ANDY$ is 20, the area of $AND'Y$ is 24, and the area of ADD' is 26, the area of $ANDD'Y$ can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

Proposed by: Andy Xu

Answer: $\boxed{285}$

Solution: Note that $ANDY$ and $AND'Y$ are isosceles trapezoids. Let $D'N$ intersect DY at P. It follows that $ANPY$ is a parallelogram. Let $x = [ANY]$ denote the area of ANY which implies that $[ANPY] = 2x$. Since $ANPY$ is a parallelogram we have $[NPY] = x$. Additionally, $[D'PY] = [AND'Y] - [ANPY] = 24 - 2x$ and similarly $[NPD] = [ANDY] - [ANPY] = 20 - 2x$.

The key observation now is that $[APD'] = [D'PY] = 24 - 2x$ since $\triangle APD'$ and $\triangle D'PY$ share the same base and height. Similarly, $[APD] = [NPD] = 20 - 2x$. Therefore, $[D'PD] = [ADD'] - [APD'] - [APD] = 26 - (24 - 2x) - (20 - 2x) = 4x - 18$. Finally, since

$$\frac{DP}{PY} = \frac{[NDP]}{[NPY]} = \frac{[D'PD]}{[D'PY]}$$

we have

$$\frac{20 - 2x}{x} = \frac{4x - 18}{24 - 2x}$$

Solving this equation yields $x = \frac{48}{7}$. Thus, $[ANDD'Y] = 26 + 2x = \frac{278}{7}$ for an answer of $\boxed{285}$.

A8. Harry wants to label the points of a regular octagon with numbers $1, 2, \ldots, 8$ and label the edges with $1, 2, \ldots, 8$. There are special rules he must follow:

- If an edge is numbered even, then the sum of the numbers of its endpoints must also be even.
- If an edge is numbered odd, then the sum of the numbers of its endpoints must also be odd.

Two octagon labelings are equivalent if they can be made equal up to rotation, but not up to reflection. If N is the number of possible octagon labelings, find the remainder when N is divided by 100.

Proposed by: Harry Kim

Answer: $\boxed{92}$

Solution: Color all even labeled points and edges blue and all odd labeled points and edges red. Notice that an edge is colored blue if its endpoints are of the same color, and is colored red if its endpoints are of different colors. Since four edges must be colored blue and four red, it follows that there must be exactly four color changes between two adjacent points. Considering rotation, we see that there are five cases: notated clockwise, the colors of the points can be BBRRBBRR, BBRRRBBR, BRRRBBBR, BRBBBRRR, or BRRBBBRR. The first case gives $\frac{1}{2}(4!)^4$ cases considering rotation. The four remaining cases give $(4!)^4$ cases, labeling the even and odd points and edges. Therefore, $N = \frac{9}{2}(4!)^4 \equiv \boxed{92} \pmod{100}$.

A9. Let $\triangle ABC$ be a triangle with $AB = 10$ and $AC = 11$. Let I be the center of the inscribed circle of $\triangle ABC$. If M is the midpoint of AI such that $BM = BC$ and

$CM = 7$, then BC can be expressed in the form $\frac{\sqrt{a}-b}{c}$ where a, b, and c are positive integers. Find $a+b+c$.

Proposed by: Andy Xu

Answer: $\boxed{622}$

Solution: After the competition ended, I was alerted that this problem was flawed since the configuration was impossible. I sincerely apologize for this mistake. Thankfully, there is no change to the rankings and results. Nevertheless, here is the intended solution.

Let D and E be points where the incircle meets AB and AC, respectively. Note that $ADIE$ is cyclic and M is the center of the circumcircle of $ADIE$. Let r be the inradius and s be the semi-perimeter of $\triangle ABC$. Power of a Point with respect to this circle yields

$$BM^2 - r^2 = BD \cdot AB = (s-b) \cdot c$$

and

$$CM^2 - r^2 = CE \cdot AC = (s-c) \cdot b$$

Subtracting the two equations yields $CM^2 - BM^2 = s(b-c) = s = \frac{a+21}{2}$. However, since $BM = a$ we know that

$$49 - a^2 = \frac{a+21}{2}$$

upon which solving gives $a = \frac{\sqrt{617}-1}{4}$ so the answer is $\boxed{622}$.

A10. Let S be a set of integers such that if a and b are in S then $3a - 2b$ is also in S. How many ways are there to construct S such that S contains exactly 4 elements in the interval $[0, 40]$?

Proposed by: Harry Kim

Answer: $\boxed{71}$

Solution: Notice that if we plot a and b in the number line, we find that $3a - 2b$ is the point 2 times the distance between a and b away from point a. Using induction, it is easy to prove that set S is either a 2-periodic set of points where the ratio of the distance between two adjacent points are $2, 1, 2, 1, \ldots$ or a set of an arithmetic sequence. Considering the 2-periodic set as an arithmetic sequence with missing points, the number of set S that contains exactly 4 elements in the interval $[0, 40]$ is equal to (number of arithmetic sequences that contains 4 numbers in the interval) + (number of arithmetic sequences that contain 5 numbers in the interval) + 3(arithmetic sequences that contain 6 numbers) + (arithmetic sequences that contain 7 numbers). Counting based on the first number of the arithmetic sequence in the interval, we get that the number of arithmetic sequences that contains 4 numbers on the interval is 28, 5 numbers is 14, 6 numbers is 8, and 7 numbers is 5. Therefore, the answer is $28 + 14 + 24 + 5 = \boxed{71}$.