# MOAA 2023 Gunga Bowl Solutions

Math Open at Andover

October 7th, 2023

G1. Find the last digit of $2023^{2023}$.

*Proposed by: Yifan Kang*

**Answer:** $\boxed{7}$

**Solution:** Note that $2023^{2023} \equiv 3^{2023} \pmod{10}$. Since $3^1 \equiv 3^5 \pmod{10}$, we find that for any $x \equiv y \pmod 4$ then $3^x \equiv 3^y \pmod{10}$. Thus,

$$3^{2023} \equiv 3^{2023-4\cdot 505} \equiv 3^3 \equiv 7 \pmod{10}$$

so our answer is $\boxed{7}$.

G2. Harry wants to put 5 identical blue books, 3 identical red books, and 1 white book on his bookshelf. If no two adjacent books may be the same color, how many distinct arrangements can Harry make?

*Proposed by: Anthony Yang*

**Answer:** $\boxed{4}$

**Solution:** Notice that there must be exactly one book between each blue book. Thus, there are four spots to put three identical red books and one white book, which yields $\binom{4}{1} = 4$ different arrangements so our answer is $\boxed{4}$.

G3. At Andover, 35% of students are lowerclassmen and the rest are upperclassmen. Given that 26% of lowerclassmen and 6% of upperclassmen take Latin, what percentage of all students take Latin? (If $a\%$ is the percentage, put $a$ as your answer).

*Proposed by: Anthony Yang*

**Answer:** $\boxed{13}$

**Solution:** For simplicity assume there are 100 students at Andover. Then, 35 students are lowerclassmen and 65 students are upperclassmen. We are given that 26% of lowerclassmen and 6% of upperclassmen take Latin, so a total of $26\% \cdot 35 + 6\% \cdot 65 = 13$ students take Latin. Thus, $\frac{13}{100} = 13\%$ of all students take Latin so our answer is $\boxed{13}$.

G4. An equilateral triangle with side length 2023 has area $A$ and a regular hexagon with side length 289 has area $B$. If $\frac{A}{B}$ can be expressed in the form $\frac{m}{n}$ where $m$ and $n$ are relatively prime, find $m + n$.

*Proposed by: Andy Xu*

**Answer:** $\boxed{55}$

**Solution:** Note that the hexagon can be seen as six equilateral triangles with side length 289. Thus, the ratio $\frac{A}{B}$ is equivalent to

$$\frac{2023^2}{289^2 \cdot 6} = \frac{49}{6}$$

so our answer is $49 + 6 = \boxed{55}$.

G5. Andy creates a 3 sided dice with a side labeled 7, a side labeled 17, and a side labeled 27. He then asks Anthony to roll the dice 3 times. The probability that the product of Anthony's rolls is greater than 2023 can be expressed in the form $\frac{m}{n}$ where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

*Proposed by: Andy Xu*

**Answer:** $\boxed{44}$

**Solution:** We will use complementary counting. Notice that in order for the product to be less than or equal to 2023, there must be at least one 7 rolled. If there is only one 7, then the other two rolls must both be 17. There are $\frac{3!}{2!} = 3$ ways to roll this. If there are two 7s, then the third roll can be either 17 or 27. There are 3 ways to roll each of these. There is only 1 way to roll three 7s. Thus, there are $3 + 3 + 3 + 1 = 10$ total ways to form a product less than or equal to 2023. Since there are $3^3 = 27$ different outcomes from three rolls, the probability of the product being greater than 2023 is $1 - \frac{10}{27} = \frac{17}{27}$ so our answer is $17 + 27 = \boxed{44}$.

G6. Andy chooses not necessarily distinct digits $G$, $U$, $N$, and $A$ such that the 5 digit number $GUNGA$ is divisible by 44. Find the least possible value of $G+U+N+G+A$.

*Proposed by: Andy Xu*

**Answer:** $\boxed{4}$

**Solution:** In order to show that $GUNGA$ is divisible by 44, we must show that $GUNGA$ is divisible by both 4 and 11. If $GUNGA$ is divisible by 11, then we must have

$$(G + N + A) - (U + G) \equiv N + A - U \equiv 0 \pmod{11}$$

or $N + A \equiv U \pmod{11}$. To minimize $N + A + U$, we have $N = A = U = 0$. To show that $GUNGA$ is divisible by 4, we need to show that $GA \equiv 0 \pmod 4$. Since we know that $A = 0$ and $G \neq 0$, the minimum value of $G$ is 2. Thus, our number $GUNGA$ is 20020, and checking we find that $\frac{20020}{44} = 455$. Our final answer is $2 + 0 + 0 + 2 + 0 = \boxed{4}$.

G7. Written in mm/dd format, a date is called *cute* if the month is divisible by the day. For example, the date 8/2 is a *cute* date because 8 is divisible by 2. Find the number of *cute* dates in a year.

*Proposed by: Andy Xu*

**Answer:** $\boxed{35}$

**Solution:** The number of *cute* dates for a given dd is equivalent to the number of multiples of dd. Let dd $= n$. Then, since mm $\leq 12$, the number of *cute* dates when dd $= n$ is $\lfloor \frac{12}{n} \rfloor$. Noticing that $\lfloor \frac{12}{n} \rfloor = 0$ when $n > 12$, our answer is just

$$\sum_{n=1}^{12} \lfloor \frac{12}{n} \rfloor = \boxed{35}.$$

G8. Let $ABCD$ be a parallelogram with area 160. Let diagonals $AC$ and $BD$ intersect at $E$. Point $P$ is on $\overline{AE}$ such that $EC = 4EP$. If line $DP$ intersects $AB$ at $F$, find the area of $BFPC$.

*Proposed by: Andy Xu*

**Answer:** $\boxed{62}$

**Solution:** Observe that $[BFPC] = [ABCD] - [ADC] - [APF]$. We know that $[ADC] = \frac{1}{2}[ABCD] = 80$, so we just need to find $[APF]$. First, notice that $\Delta CPD \sim \Delta APF$ since all sides are parallel to each other. We have

$$\frac{[CPD]}{[ADC]} = \frac{CP}{CA} = \frac{5}{8}$$

so $[DPC] = \frac{5}{8} \cdot 80 = 50$. Now, notice that

$$\frac{[APF]}{[CPD]} = \frac{AP^2}{CP^2} = \frac{9}{25}$$

so $[APF] = \frac{9}{25} \cdot 50 = 18$. Thus, we get

$$[BFPC] = [ABCD] - [ADC] - [APF] = 160 - 80 - 18 = 62$$

so our answer is $\boxed{62}$.

G9. Real numbers $x$ and $y$ satisfy

$$xy + \frac{x}{y} = 3$$
$$\frac{1}{x^2y^2} + \frac{y^2}{x^2} = 4$$

If $x^2$ can be expressed in the form $\frac{a+\sqrt{b}}{c}$ for integers $a$, $b$, and $c$. Find $a + b + c$.

*Proposed by: Andy Xu*

**Answer:** $\boxed{40}$

**Solution:** Dividing the first equation by $x$ yields $y + \frac{1}{y} = \frac{3}{x}$. Multiplying the second equation by $x^2$ yields $\frac{1}{y^2} + y^2 = 4x^2$. Notice that $(y + \frac{1}{y})^2 = y^2 + 2 + \frac{1}{y^2} = \frac{9}{x^2}$. Setting $y^2 + \frac{1}{y^2} = z$ gives the following equations:

$$z = 4x^2$$
$$z + 2 = \frac{9}{x^2}$$

Multiplying the two equations together gives

$$z(z + 2) = 36 \Rightarrow z^2 + 2z - 36 = 0$$

so $z = -1 \pm \sqrt{37}$ from quadratic formula. However, since $x$ is real and $z = 4x^2$ we must have $z > 0$ so $z = -1 + \sqrt{37}$. Since we are looking for $x^2$, we get

$$x^2 = \frac{z}{4} = \frac{-1 + \sqrt{37}}{4}$$

so our final answer is $-1 + 37 + 4 = \boxed{40}$.

G10. A number is called *winning* if it can be expressed in the form $\frac{a}{20} + \frac{b}{23}$ where $a$ and $b$ are positive integers. How many *winning* numbers are less than 1?

*Proposed by: Andy Xu*

**Answer:** $\boxed{209}$

**Solution:** We want to find all numbers that can be expressed in the form

$$\frac{a}{20} + \frac{b}{23} < 1$$

where $a$ and $b$ are positive integers. Multiplying both sides by 460 gives

$$23a + 20b < 460.$$

Now consider the graph of this inequality. We wish to find all lattice points within the region bounded by $23x + 20y = 460$, the $x$-axis, and the $y$-axis, which we will do using Pick's theorem. Since the area of this region is $\frac{1}{2} \cdot 23 \cdot 20 = 230$ and there are 44 points on the boundary of this region, we get

$$230 = I + \frac{44}{2} - 1 \Rightarrow I = 209$$

where $I$ represents the number of lattice points within the region. Since $I$ is what we want to find, our answer is $\boxed{209}$.

G11. Let $s(n)$ denote the sum of the digits of $n$ and let $p(n)$ be the product of the digits of $n$. Find the smallest integer $k$ such that $s(k)+p(k) = 49$ and $s(k+1)+p(k+1) = 68$.

*Proposed by: Anthony Yang*

**Answer:** $\boxed{292}$

**Solution:** Note that if there is carryover from $k$ to $k + 1$, then $p(k + 1) = 0$ and $s(k + 1) < s(k)$. Thus, we cannot have any carryover from $k$ to $k + 1$. If there is no carryover, then $s(k + 1) = s(k) + 1$ so we now have the following equations:

$$s(k) + p(k) = 49$$
$$s(k) + 1 + p(k + 1) = 68$$

Subtracting the first equation from the second yields

$$p(k + 1) - p(k) = 18.$$

Now notice that if $k$ is an $n$-digit number and there is no carryover from $k$ to $k + 1$, then $p(k + 1) - p(k)$ is equivalent to the product of the first $n - 1$ digits of $k$. For example, $p(125) - p(124) = (1 \cdot 2 \cdot 5) - (1 \cdot 2 \cdot 4) = 1 \cdot 2 \cdot (5 - 4) = 1 \cdot 2 = 2$. Thus, we now want to find the smallest $(n - 1)$-digit integer $k'$ such that the product of the digits of $k'$ is 18. This just yields $k' = 29$, so now we have $k = \overline{29b}$ for some digit $b$. Plugging this into our original equation gives

$$s(\overline{29b}) + p(\overline{29b}) = 2 + 9 + b + 18b = 11 + 19b = 49$$

which means $b = 2$ so our final answer is $\boxed{292}$.

G12. Andy is planning to flip a fair coin 10 times. Among the 10 flips, Valencia randomly chooses one flip to exchange Andy's fair coin with her special coin which lands on heads with a probability of $\frac{1}{4}$. If the coin is exchanged in a certain flip, then that flip, along with all following flips will be performed with the special coin. The expected number of heads Andy flips can be expressed as $\frac{m}{n}$ where $m$ and $n$ are positive integers. Find $m+n$.

*Proposed by: Andy Xu*

**Answer:** $\boxed{37}$

**Solution:** Assume that Valencia chooses to exchange her special coin with Andy's fair coin on the $n+1^{\text{th}}$ flip, where $0 \le n \le 9$. In this case, Andy's coin is flipped $n$ times while Valencia's coin is flipped $10-n$ times. Thus, the expected number of heads flipped when Valencia picks the $n+1^{\text{th}}$ coin is

$$\frac{n}{2} + \frac{10-n}{4} = \frac{10+n}{4}.$$

Since there are ten different cases and each one is equally likely to be picked, the expected number of heads flipped over all cases is

$$\frac{1}{10}\sum_{n=0}^{9} \frac{10+n}{4} = \frac{29}{8}$$

so our answer is $29+8 = \boxed{37}$.

G13. Let $\alpha$, $\beta$ and $\gamma$ be the roots of the polynomial $2023x^3 - 2023x^2 - 1$. Find

$$\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3}$$

.

*Proposed by: Andy Xu*

**Answer:** $\boxed{6069}$

**Solution 1:** Notice that

$$\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} = (\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma})^3 - 3(\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma})(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}) + \frac{3}{\alpha\beta\gamma}.$$

This can further be rewritten as

$$\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} = (\frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma})^3 - 3(\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma})(\frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}) + \frac{3}{\alpha\beta\gamma}.$$

Now using Vieta's, we get the following equations:

$$\begin{aligned} &\alpha+\beta+\gamma = 1 \\ &\alpha\beta + \alpha\gamma + \beta\gamma = 0 \\ &\alpha\beta\gamma = \tfrac{1}{2023} \end{aligned}$$

Plugging each of these in gives

$$\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} = 0^3 - 3 \cdot \frac{1}{\frac{1}{2023}} \cdot 0 + \frac{3}{\frac{1}{2023}} = 2023 \cdot 3 = 6069$$

so our answer is $\boxed{6069}$.

**Solution 2:** Note the polynomial with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$, and $\frac{1}{\gamma}$ is

$$P(x) = x^3 + 2023x - 2023$$

Observe that the sum of the roots is

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=0$$

and it is well known consequence that

$$\left(\frac{1}{\alpha}\right)^3+\left(\frac{1}{\beta}\right)^3+\left(\frac{1}{\gamma}\right)^3=3\cdot\frac{1}{\alpha}\cdot\frac{1}{\beta}\cdot\frac{1}{\gamma}=\frac{3}{\alpha\beta\gamma}=\frac{3}{\frac{1}{2023}}=\boxed{6069}$$

G14. Let $N$ be the number of ordered triples of 3 positive integers $(a,b,c)$ such that $6a$, $10b$, and $15c$ are all perfect squares and $abc=210^{210}$. Find the number of divisors of $N$.

*Proposed by: Andy Xu*

**Answer:** $\boxed{640}$

**Solution:** Since $210=2\cdot 3\cdot 5\cdot 7$ the only possible prime factors of $a$, $b$, and $c$ are 2, 3, 5, and 7. Let $a=2^{a_1}3^{a_2}5^{a_3}7^{a_4}$, $b=2^{b_1}3^{b_2}5^{b_3}7^{b_4}$, and $c=2^{c_1}3^{c_2}5^{c_3}7^{c_4}$ where each of $a_i,b_i,c_i$ for $1\le i\le 4$ are nonnegative integers. Then, if $6a=2^{a_1+1}3^{a_2+1}5^{a_3}7^{a_4}$ is a perfect square, we must have

$$a_1\equiv a_2\equiv 1 \pmod 2$$
$$a_3\equiv a_4\equiv 0 \pmod 2$$

Doing the same for $10b$ and $15c$ yields

$$a_1\equiv a_2\equiv b_1\equiv b_3\equiv c_2\equiv c_3\equiv 1 \pmod 2$$
$$a_3\equiv a_4\equiv b_2\equiv b_4\equiv c_1\equiv c_4\equiv 0 \pmod 2$$

Now we must have

$$a_1+b_1+c_1=210.$$

Letting $a_1=2a_1'+1$, $b_1=2b_1'+1$, and $c_1=2c_1'$ gives

$$(2a_1'+1)+(2b_1'+1)+2c_1'=210\Rightarrow a_1'+b_1'+c_1'=104.$$

Since the only requirement for $a_1'$, $b_1'$, and $c_1'$ is for them to be nonnegative integers, there are $\binom{106}{2}$ triples $(a_1,b_1,c_1)$ from stars and bars. Similarly, we find that there are $\binom{106}{2}$ triples that work for $(a_2,b_2,c_2)$ and $(a_3,b_3,c_3)$. Finally, we check

$$a_4+b_4+c_4=210$$

Letting $a_4=2a_4'$, $b_4=2b_4'$, and $c_4=2c_4'$, we get

$$2a_4'+2b_4'+2c_4'=210\Rightarrow a_4'+b_4'+c_4'=105$$

which yields $\binom{107}{2}$ triples $(a_4,b_4,c_4)$ that work. Thus, we have

$$N=\binom{106}{2}^3\binom{107}{2}=3^3\cdot 5^3\cdot 7^3\cdot 53^4\cdot 107^1$$

so the number of divisors of $N$ is $(3+1)(3+1)(3+1)(4+1)(1+1)=\boxed{640}$.

G15. Triangle $ABC$ has $AB = 5$, $BC = 7$, $CA = 8$. Let $M$ be the midpoint of $BC$ and let points $P$ and $Q$ lie on $AB$ and $AC$ respectively such that $MP \perp AB$ and $MQ \perp AC$. If $H$ is the orthocenter of $\triangle APQ$ then the area of $\triangle HPM$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ where $a$ and $c$ are relatively prime positive integers and $b$ is square-free. Find $a + b + c$.

*Proposed by: Harry Kim*

**Answer:** $\boxed{26}$

**Solution:** Notice that since $MQ \perp AC$ and $PH \perp AC$, and $MP \perp AB$ and $QH \perp AB$, we have $MQ \parallel HP$ and $MP \parallel QH$ so $MPHQ$ is a parallelogram. Now let $D$ and $E$ be the feet of the altitudes from $B$ and $P$ to $AC$ respectively, and let $F$ be the foot of the altitude from $M$ to $HP$. Observe that since $MPHQ$ is a parallelogram, $FM = EQ$ and $HP = MQ$. Thus, we have

$$[HPM] = \frac{1}{2} \cdot HP \cdot FM = \frac{1}{2} \cdot MQ \cdot EQ.$$

Since $M$ is the midpoint of $BC$ we know that $MQ = \frac{BD}{2}$, and since $BD$ is an altitude we know $BD = \frac{2[ABC]}{AC}$. From Heron's formula we get $[ABC] = 10\sqrt{3}$ so

$$BD = \frac{2 \cdot 10\sqrt{3}}{8} = \frac{5\sqrt{3}}{2} \Rightarrow MQ = \frac{5\sqrt{3}}{4}.$$

We can quickly see that $\Delta BAD$ is a $30 - 60 - 90$ right triangle, so we have $\angle A = 60^\circ \Rightarrow \angle HQA = 90^\circ - \angle A = 30^\circ$. Thus, $\Delta QHE$ is also a $30 - 60 - 90$ triangle. Notice that $HQ = MP$. Since $MP$ is half the altitude from $C$ to $AB$ we have $MP = 2\sqrt{3} = HQ$. Then, from $\Delta QHE$ we have $EQ = 3$ so

$$[HPM] = \frac{1}{2} \cdot \frac{5\sqrt{3}}{4} \cdot 3 = \frac{15\sqrt{3}}{8}$$

so our final answer is $15 + 3 + 8 = \boxed{26}$.

G16. Compute the sum

$$\frac{\varphi(50!)}{\varphi(49!)} + \frac{\varphi(51!)}{\varphi(50!)} + \cdots + \frac{\varphi(100!)}{\varphi(99!)}$$

where $\varphi(n)$ returns the number of positive integers less than $n$ that are relatively prime to $n$.

*Proposed by: Andy Xu*

**Answer:** $\boxed{3815}$

**Solution:** Consider an integer $k$ such that $n = p_1^{e_1} p_2^{e_2} \dots, p_m^{e_m}$ where $p_1, p_2, \dots, p_m$ are distinct prime numbers and $e_1, e_2, \dots, e_m$ are positive integers. Then,

$$\varphi(k) = \frac{p_1 - 1}{p_1} p_1^{e_1} \cdot \frac{p_2 - 1}{p_2} p_2^{e_2} \cdot \dots \cdot \frac{p_m - 1}{p_m} p_m^{e_m}$$

Now consider $\frac{\varphi(n+1!)}{\varphi(n!)}$ where $n+1$ is not prime. Let $n+1 = p_{n_1}^{e_{n_1}} p_{n_2}^{e_{n_2}} \dots p_{n_j}^{e_{n_j}}$. Notice that $\{n_1, n_2, \dots, n_j\} \in \{1, 2, \dots, m\}$, which means that

$$\frac{\varphi(n+1!)}{\varphi(n!)} = \frac{\varphi(n! \cdot (n+1))}{\varphi(n!)} = \frac{\varphi(n! \cdot p_{n_1}^{e_{n_1}} p_{n_2}^{e_{n_2}} \dots p_{n_j}^{e_{n_j}})}{\varphi(n!)} = p_{n_1}^{e_{n_1}} p_{n_2}^{e_{n_2}} \dots p_{n_j}^{e_{n_j}} = n+1$$

Now consider $\frac{\varphi(n+1!)}{\varphi(n!)}$ where $n+1$ is prime. Then, since $n+1 \mid (n+1)!$ and $n+1 \nmid n!$, we have

$$\frac{\varphi(n+1!)}{\varphi(n!)} = \frac{\varphi(n! \cdot (n+1))}{\varphi(n!)} = \frac{(n+1)-1}{n+1}(n+1) = n.$$

Therefore we get

$$\frac{\varphi(50!)}{\varphi(49!)} + \frac{\varphi(51!)}{\varphi(50!)} + \cdots + \frac{\varphi(100!)}{\varphi(99!)} = 50 + 51 + \ldots + 100 - n_p$$

where $n_p$ represents the number of primes $p$ such that $50 \leq p \leq 100$. Counting, we get $n_p = 10$ so our answer is $50 + 51 + \ldots + 100 - 10 = 3825 - 10 = \boxed{3815}$.

G17. Call a polynomial with real roots *n-local* if the greatest difference between any pair of its roots is $n$. Let $f(x) = x^2 + ax + b$ be a 1-*local* polynomial with distinct roots such that $a$ and $b$ are non-zero integers. If $f(f(x))$ is a 23-*local* polynomial, find the sum of the roots of $f(x)$.

*Proposed by: Anthony Yang*

**Answer:** $\boxed{263}$

**Solution:** Let the roots of $f(x)$ be $r_1$ and $r_2$. Then, since $f(x)$ is a *1-local* polynomial, we have $|r_2 - r_1| = 1$. Additionally, if $a = -(r_1 + r_2)$ and $b = r_1 r_2$ are both integers, then $r_1$ and $r_2$ must also be integers. WLOG let $r_2 = r_1 + 1$. Now, we can write

$$f(x) = x^2 - (r_1 + r_2)x + r_1 r_2 = (x - r_1)(x - r_2) = (x - r_1)(x - (r_1 + 1)).$$

Observe that $f(f(x))$ can be expressed as $(f(x) - r_1)(f(x) - (r_1 + 1))$ or

$$((x - r_1)(x - (r_1 + 1)) - r_1)((x - r_1)(x - (r_1 + 1)) - (r_1 + 1)).$$

Thus, the roots of $f(f(x))$ must satisfy

$$((x - r_1)(x - (r_1 + 1)) - r_1)((x - r_1)(x - (r_1 + 1)) - (r_1 + 1)) = 0$$

meaning that we have either of the following equations:

$$(x - r_1)(x - (r_1 + 1)) - r_1 = 0$$
$$(x - r_1)(x - (r_1 + 1)) - (r_1 + 1) = 0$$

We find that the roots of $(x - r_1)(x - (r_1 + 1)) - r_1 = 0$ are $\frac{(2r_1+1) \pm \sqrt{4r_1+1}}{2}$ and the roots of $(x - r_1)(x - (r_1 + 1)) - (r_1 + 1) = 0$ are $\frac{(2r_1+1) \pm \sqrt{4r_1+5}}{2}$ by expanding and using quadratic formula. Now, note that the greatest possible absolute difference between any pair of roots is

$$\frac{(2r_1 + 1) + \sqrt{4r_1 + 5}}{2} - \frac{(2r_1 + 1) - \sqrt{4r_1 + 5}}{2} = \sqrt{4r_1 + 5}.$$

Thus, we have

$$\sqrt{4r_1 + 5} = 23$$

so $r_1 = \frac{23^2 - 5}{4} = 131$. Our final answer is $r_1 + (r_1 + 1) = 131 + 132 = \boxed{263}$.

G18. Triangle $\triangle ABC$ is isosceles with $AB = AC$. Let the incircle of $\triangle ABC$ intersect $BC$ and $AC$ at $D$ and $E$ respectively. Let $F \neq A$ be the point such that $DF = DA$ and $EF = EA$. If $AF = 8$ and the circumradius of $\triangle AED$ is 5, find the area of $\triangle ABC$.

*Proposed by: Anthony Yang and Andy Xu*

**Answer:** $\boxed{60}$

**Solution:** Extend $EF$ to intersect $AD$ at $L$ and let $DF$ intersect $AC$ at $M$. Now let $\angle AEF = \alpha$. Then, from isosceles triangles we have $\angle EFA = \angle EAF = 90^\circ - \frac{\alpha}{2}$ and $\angle DAF = \angle DFA \Rightarrow \angle DAE = \angle DFE$. Notice that since $DE$ bisects $\angle ADF$ we have $\Delta DAE \cong \Delta DFE$. Since $\angle AEL = \angle FEM$, we have $\Delta AEL \cong \Delta FEM$ so $EL = EM$ and $AD - AL = FD - FM \Rightarrow DL = DM$. Thus, $ELDM$ is a kite. We know that $ED$ bisects $\angle LEM = \alpha$ so $\angle MED = \frac{\alpha}{2}$. Additionally, from equal tangents we have

$$CE = CD \Rightarrow \angle CDE = \angle CED = \frac{\alpha}{2} \Rightarrow \angle DCE = 180^\circ - \alpha$$

so $\angle DAC = 180^\circ - \angle ADC - \angle ACD = \alpha - 90^\circ$. Since $\angle AEL = 180^\circ - \alpha$, we have $\angle ALE = 180^\circ - \angle LAE - \angle AEL = 90^\circ$ which means that $LE \parallel DC$. Now let the incircle of $\Delta ABC$ intersect $AB$ at $N$. First notice that $N, L, E$, and $F$ are collinear. Furthermore, $\angle NDL = \angle EDL = 90^\circ - \frac{\alpha}{2}$ and $\angle NAL = \angle EAL = \alpha - 90^\circ$. Since $DE$ bisects $\angle ADF$ we know $\angle EDF = \angle EDA = 90^\circ - \frac{\alpha}{2}$ so we have

$$\angle NDF = \angle NDA + \angle ADE + \angle EDF = 270^\circ - \frac{3\alpha}{2}.$$

Now notice that $\angle NAD = \angle DAC = \alpha - 90^\circ$ so

$$\angle NAF = \angle NAD + \angle DAE + \angle EAF = \frac{3\alpha}{2} - 90^\circ$$

so $\angle NDF + \angle NAF = 180^\circ$ so $NAFD$ is cyclic. Since $\Delta AED \cong \Delta AND$, the circumradius of $\Delta AND$ and subsequently $\Delta DAF$ must be 5. Let $DA = DF = x$. Since the area of a triangle is equivalent to $\frac{abc}{4R}$ we have

$$[\Delta DAF] = 4\sqrt{x^2 - 16} = \frac{8x^2}{4 \cdot 5}$$

which simplifies to $x^4 - 100x^2 + 1600 = 0$. Solving gives $x^2 = 20, 80$ so $x = 2\sqrt{5}$, $4\sqrt{5}$. However, since $\angle ADF < \angle ADC = 90^\circ$, we must have $x = 4\sqrt{5}$. Let the altitude from $D$ meet $AF$ at $K$. $AK = \sqrt{AD^2 - AK^2} = 8$ and $\Delta DAK \sim \Delta AEK$ so we have

$$AE = \frac{AK}{DK} \cdot AD = 2\sqrt{5}$$

Furthermore, since $\Delta DLE \sim \Delta DKA$ and $DE = AK - DK = 6$ we have

$$LE = \frac{DE}{DA} \cdot AK = \frac{6\sqrt{5}}{5}.$$

Since $\Delta ALE$ is a right triangle we get $AL = \frac{8\sqrt{5}}{5}$ and since $\Delta ALE \sim \Delta ADC$ we get

$$DC = \frac{AD}{AL} \cdot LE = 3\sqrt{5}$$

so our answer is $\frac{3\sqrt{5} \cdot 4\sqrt{5}}{2} \cdot 2 = \boxed{60}$.

G19. Compute the remainder when $\binom{205}{101}$ is divided by $101 \times 103$.

*Proposed by: Brandon Xu*

**Answer:** $\boxed{5355}$

**Solution:** We use a clever application of Vandermonde's identity, which states:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} \tag{1}$$

Letting $m, n, r$ be $101, 104, 101$ respectively gives:

$$\binom{205}{101} = \binom{101+104}{101} = \sum_{k=0}^{101} \binom{101}{k}\binom{104}{101-k}$$

Note that for all $1 \le k \le 100$, $\binom{101}{k}\binom{104}{101-k} \equiv 0 \pmod{101}$. Thus, the sum reduces to

$$N \equiv \binom{101}{0}\binom{104}{101} + \binom{101}{101}\binom{104}{0} \equiv 2 \pmod{101}$$

Similarly, for all $0 \le k \le 99$, $\binom{101}{k}\binom{104}{101-k} \equiv 0 \pmod{103}$. In this case, the sum reduces to

$$N \equiv \binom{101}{100}\binom{104}{1} + \binom{101}{101}\binom{104}{0} \equiv -2+1 \equiv -1 \pmod{103}$$

We have $N \equiv 2 \pmod{101}$ and $N \equiv -1 \pmod{103}$. Combining these two relations yields $N \equiv 5355 \pmod{101 \times 103}$ for an answer of $\boxed{5355}$.

G20. Big Bad Brandon is assigning groups of his Big Bad Burglars to attack 7 different towers. Each Burglar can only belong to one attack group and Brandon takes over a tower if the number of Burglars attacking the tower strictly exceeds the number of knights guarding it. He knows there the total number of knights guarding the towers is 99 but does not know the exact number of knights guarding each tower. What is the minimum number of Burglars that Brandon needs to guarantee he can take over at least 4 of the 7 towers?

*Proposed by: Eric Wang*

**Answer:** $\boxed{175}$

**Solution:** Let the number of burglars assigned to the $i$th tower be $b_i$. WLOG let $b_1 \le b_2 \le \cdots \le b_7$. Also, let the number of knights assigned to the $i$th tower be $k_i$.

Notice that Brandon fails if and only if $k_i \ge b_i$ for at least 4 values of $i$. The minimum number of knights needed for this to happen would be

$$b_1 + b_2 + b_2 + b_4$$

which occurs when $k_i = b_i$ for $1 \le i \le 4$. Therefore to guarantee Brandon will succeed it is necessary and sufficient that $b_1 + b_2 + b_3 + b_4$ is greater than the total number of knights, that is,

$$b_1 + b_2 + b_3 + b_4 \ge 100.$$

After that, notice that $b_7 \geq b_6 \geq b_5 \geq b_4$, so the total number of burglars is

$$\sum_{i=1}^{7} \geq (b_1 + b_2 + b_3 + b_4) + 3b_4 \geq 100 + 3b_4,$$

and the minimum value of $b_4$ is when $b_1 = b_2 = b_3 = b_4 = 25$, so the minimum number of Burglars that Brandon needs is $100 + 3 \cdot 25 = \boxed{175}$.

G21. In obtuse triangle $ABC$ where $\angle B > 90^\circ$ let $H$ and $O$ be its orthocenter and circumcenter respectively. Let $D$ be the foot of the altitude from $A$ to $HC$ and $E$ be the foot of the altitude from $B$ to $AC$ such that $O, E, D$ lie on a line. If $OC = 8$ and $OE = 4$, find the area of triangle $HAB$.

*Proposed by: Harry Kim*

**Answer:** $\boxed{32}$

**Solution:** Observe that $\angle AEC = \angle ADC = 90^\circ$ so $AECD$ is cyclic. Therefore, $\angle OEB = \angle CED = \angle CAD = \frac{\angle BOC}{2} = 90^\circ - \angle OBE$ so $\angle BOE = 90^\circ$. Thus, $BE = \sqrt{8^2 + 4^2} = 4\sqrt{5}$. Let $M$ be the midpoint of $BC$. Then $OM = \frac{32}{4\sqrt{5}} = \frac{8}{\sqrt{5}}$. It is well known that $HA = 2OM = \frac{16}{\sqrt{5}}$. Thus,

$$[\triangle HAB] = \frac{1}{2} BE \cdot HA = \frac{1}{2} \cdot 4\sqrt{5} \cdot \frac{16}{\sqrt{5}} = \boxed{32}.$$

G22. Harry the knight is positioned at the origin of the Cartesian plane. In a "knight hop", Harry can move from the point $(i, j)$ to a point with integer coordinates that is a distance of $\sqrt{5}$ away from $(i, j)$. What is the number of ways that Harry can return to the origin after 6 knight hops?

*Proposed by: Harry Kim*

**Answer:** $\boxed{5840}$

**Solution:** We can separate into four cases: when the total positive $x$ movement is 6, 5, 4, and 3. By symmetry with the positive $y$ movement, the first and the fourth case yield the same number and the second and the third case yield the same number as well. Therefore, we consider the first two cases.

(i) Total positive $x$ movement: 6

The $x$ movements must be a permutation of $(2, 2, 2, -2, -2, -2)$ and the $y$ movements must be a permutation of $(1, 1, 1, -1, -1, -1)$. Therefore, $\binom{6}{3} \cdot \binom{6}{3} = 400$.

(ii) Total positive $x$ movement: 5

The $x$ movements must be a permutation of $(2, 2, 1, -2, -2, -1)$. Then the $y$ movements can be a permutation of $(2, 2, -1, -1, -1, -1)$, $(-2, -2, 1, 1, 1, 1)$, and $(2, 1, 1, -1, -1, -2)$. This gives $\frac{6!}{2!2!} \cdot (1 + 1 + 2\binom{4}{2}) = 2520$ possible ways.

Therefore, the answer is $2(400 + 2520) = \boxed{5840}$.

G23. For every positive integer $n$ let

$$f(n) = \frac{n^4 + n^3 + n^2 - n + 1}{n^6 - 1}$$

Given

$$\sum_{n=2}^{20} f(n) = \frac{a}{b}$$

for relatively prime positive integers $a$ and $b$, find the sum of the prime factors of $b$.

*Proposed by: Harry Kim*

**Answer:** $\boxed{435}$

**Solution:** Notice that

$$\begin{aligned} f(n) = \frac{n^4+n^3+n^2-n+1}{n^6-1} &= \frac{n^4+n^2+1}{n^6-1} + \frac{n(n^2-1)}{n^6-1} \\ = \frac{1}{n^2-1} + \frac{n}{n^4+n^2+1} &= \frac{1}{(n+1)(n-1)} + \frac{n}{(n^2+n+1)(n^2-n+1)} \end{aligned}$$

Observe that

$$\begin{aligned} \sum_{n=2}^{20} \frac{1}{(n+1)(n-1)} &= \frac{1}{2} \cdot \sum_{n=2}^{20} \left( \frac{1}{n-1} - \frac{1}{n+1} \right) \\ &= \frac{1}{2}\left( \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \cdots \left(\frac{1}{19} - \frac{1}{21}\right) \right) \\ &= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{20} - \frac{1}{21}\right) \end{aligned}$$

and

$$\begin{aligned} \sum_{n=2}^{20} \frac{n}{(n^2+n+1)(n^2-n+1)} &= \frac{1}{2} \cdot \sum_{n=2}^{20} \left( \frac{1}{n^2-n+1} - \frac{1}{n^2+n+1} \right) \\ &= \frac{1}{2}\left( \left(\frac{1}{3} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{13}\right) + \cdots \left(\frac{1}{381} - \frac{1}{421}\right) \right) \\ &= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{421}\right) \end{aligned}$$

Then the desired value is

$$\frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{20} - \frac{1}{21} - \frac{1}{421}\right) = \frac{102163}{117880}$$

Since $117880 = 2^3 \times 5 \times 7 \times 421$, the answer is $2 + 5 + 7 + 421 = \boxed{435}$.

G24. Circle $\omega$ is inscribed in acute triangle $ABC$. Let $I$ denote the center of $\omega$, and let $D, E, F$ be the points of tangency of $\omega$ with $BC, CA, AB$ respectively. Let $M$ be the midpoint of $BC$, and $P$ be the intersection of the line through $I$ perpendicular to $AM$ and line $EF$. Suppose that $AP = 9$, $EC = 2EA$, and $BD = 3$. Find the sum of all possible perimeters of $\triangle ABC$.

*Proposed by: Harry Kim*

**Answer:** $\boxed{57}$

**Solution:** Let $X$ be the intersection of $AM$ and $EF$. It is well known that the points $D, I, X$ are collinear. Observe that $AX \perp PI$ and $AI \perp PX$ so $X$ is the orthocenter of triangle $AIP$. Hence, $DX \perp AP$ and therefore $AP \parallel BC$. Now let lines $EF$ and $BC$ intersect at $Q$. Using the length conditions, let $AE = AF = x$ and $CD = CE = 2x$. Notice that $\triangle AFP \sim \triangle BFQ$ so $QB = \frac{27}{x}$. Using Menelaus theorem at $ABC - EFQ$, we obtain

$$\frac{x}{3} \cdot \frac{27}{2x^2 + 3x + 27} \cdot 2 = 1$$

Rearranging, we get $2x^2 - 15x + 27 = (2x - 9)(x - 3) = 0$ so $x = \frac{9}{2}$ or $x = 3$. When $x = \frac{9}{2}$, the perimeter of $\triangle ABC$ is $6x + 6 = 33$ and when $x = 3$, the perimeter is $6x + 6 = 24$. Therefore, the answer is $33 + 24 = \boxed{57}$.