MOAA 2023: Gunga Bowl

October 7th, 2023

Gunga Bowl Problems

Gunga Bowl Set 1

G1. **[8]** Find the last digit of 2023^{2023}.

G2. **[8]** Harry wants to put 5 identical blue books, 3 identical red books, and 1 white book on his bookshelf. If no two adjacent books may be the same color, how many distinct arrangements can Harry make?

G3. **[8]** At Andover, 35% of students are lowerclassmen and the rest are upperclassmen. Given that 26% of lowerclassmen and 6% of upperclassmen take Latin, what percentage of all students take Latin? (If $a\%$ is the percentage, put a as your answer).

Gunga Bowl Set 2

G4. **[10]** An equilateral triangle with side length 2023 has area A and a regular hexagon with side length 289 has area B. If $\frac{A}{B}$ can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime, find $m+n$.

G5. **[10]** Andy creates a 3 sided dice with a side labeled 7, a side labeled 17, and a side labeled 27. He then asks Anthony to roll the dice 3 times. The probability that the product of Anthony's rolls is greater than 2023 can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

G6. **[10]** Andy chooses not necessarily distinct digits G, U, N, and A such that the 5 digit number $GUNGA$ is divisible by 44. Find the least possible value of $G+U+N+G+A$.

Gunga Bowl Set 3

G7. **[12]** Written in mm/dd format, a date is called *cute* if the month is divisible by the day. For example, the date 8/2 is a *cute* date because 8 is divisible by 2. Find the number of *cute* dates in a year.

G8. **[12]** Let $ABCD$ be a parallelogram with area 160. Let diagonals AC and BD intersect at E. Point P is on $\overline{AE}$ such that $EC = 4EP$. If line DP intersects AB at F, find the area of $BFPC$.

G9. **[12]** Real numbers x and y satisfy

$$xy + \frac{x}{y} = 3$$

$$\frac{1}{x^2y^2} + \frac{y^2}{x^2} = 4$$

If x^2 can be expressed in the form $\frac{a+\sqrt{b}}{c}$ for integers a, b, and c. Find $a+b+c$.

Gunga Bowl Set 4

G10. **[14]** A number is called *winning* if it can be expressed in the form $\frac{a}{20} + \frac{b}{23}$ where a and b are positive integers. How many *winning* numbers are less than 1?

G11. **[14]** Let $s(n)$ denote the sum of the digits of n and let $p(n)$ be the product of the digits of n. Find the smallest integer k such that $s(k) + p(k) = 49$ and $s(k+1) + p(k+1) = 68$.

G12. **[14]** Andy is planning to flip a fair coin 10 times. Among the 10 flips, Valencia randomly chooses one flip to exchange Andy's fair coin with her special coin which lands on heads with a probability of $\frac{1}{4}$. If the coin is exchanged in a certain flip, then that flip, along with all following flips will be performed with the special coin. The expected number of heads Andy flips can be expressed as $\frac{m}{n}$ where m and n are positive integers. Find $m+n$.

Gunga Bowl Set 5

G13. **[16]** Let α, β and γ be the roots of the polynomial $2023x^3 - 2023x^2 - 1$. Find

$$\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3}$$

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G14. **[16]** Let N be the number of ordered triples of 3 positive integers (a, b, c) such that $6a$, $10b$, and $15c$ are all perfect squares and $abc = 210^{210}$. Find the number of divisors of N.

G15. **[16]** Triangle ABC has $AB = 5$, $BC = 7$, $CA = 8$. Let M be the midpoint of BC and let points P and Q lie on AB and AC respectively such that $MP \perp AB$ and $MQ \perp AC$. If H is the orthocenter of $\triangle APQ$ then the area of $\triangle HPM$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ where a and c are relatively prime positive integers and b is square-free. Find $a + b + c$.

Gunga Bowl Set 6

G16. **[18]** Compute the sum

$$\frac{\varphi(50!)}{\varphi(49!)} + \frac{\varphi(51!)}{\varphi(50!)} + \cdots + \frac{\varphi(100!)}{\varphi(99!)}$$

where $\varphi(n)$ returns the number of positive integers less than n that are relatively prime to n.

G17. **[18]** Call a polynomial with real roots *n-local* if the greatest difference between any pair of its roots is n. Let $f(x) = x^2 + ax + b$ be a 1-*local* polynomial with distinct roots such that a and b are non-zero integers. If $f(f(x))$ is a 23-*local* polynomial, find the sum of the roots of $f(x)$.

G18. **[18]** Triangle $\triangle ABC$ is isosceles with $AB = AC$. Let the incircle of $\triangle ABC$ intersect BC and AC at D and E respectively. Let $F \neq A$ be the point such that $DF = DA$ and $EF = EA$. If $AF = 8$ and the circumradius of $\triangle AED$ is 5, find the area of $\triangle ABC$.

Gunga Bowl Set 7

G19. **[20]** Compute the remainder when $\binom{205}{101}$ is divded by 101×103.

G20. **[20]** Big Bad Brandon is assigning groups of his Big Bad Burglars to attack 7 different towers. Each Burglar can only belong to one attack group and Brandon takes over a tower if the number of Burglars attacking the tower strictly exceeds the number of knights guarding it. He knows there the total number of knights guarding the towers is 99 but does not know the exact number of knights guarding each tower. What is the minimum number of Burglars that Brandon needs to guarantee he can take over at least 4 of the 7 towers?

G21. **[20]** In obtuse triangle ABC where $\angle B > 90^\circ$ let H and O be its orthocenter and circumcenter respectively. Let D be the foot of the altitude from A to HC and E be the foot of the altitude from B to AC such that O, E, D lie on a line. If $OC = 8$ and $OE = 4$, find the area of triangle HAB.

Gunga Bowl Set 8

G22. **[22]** Harry the knight is positioned at the origin of the Cartesian plane. In a "knight hop", Harry can move from the point (i, j) to a point with integer coordinates that is a distance of $\sqrt{5}$ away from (i, j). What is the number of ways that Harry can return to the origin after 6 knight hops?

G23. **[22]** For every positive integer n let

$$f(n) = \frac{n^4 + n^3 + n^2 - n + 1}{n^6 - 1}$$

Given

$$\sum_{n=2}^{20} f(n) = \frac{a}{b}$$

for relatively prime positive integers a and b, find the sum of the prime factors of b.

G24. **[22]** Circle ω is inscribed in acute triangle ABC. Let I denote the center of ω, and let D, E, F be the points of tangency of ω with BC, CA, AB respectively. Let M be the midpoint of BC, and P be the intersection of the line through I perpendicular to AM and line EF. Suppose that $AP = 9$, $EC = 2EA$, and $BD = 3$. Find the sum of all possible perimeters of $\triangle ABC$.

Gunga Bowl Set 9

This set consists of three estimation problems, with scoring schemes described.

G25. **[30]** Estimate N, the total number of participants (in person and online) at MOAA this year.

An estimate of e gets a total of $\max\left(0, \left\lfloor 150\left(1 - \frac{|N-e|}{N}\right)\right\rfloor - 120\right)$ points.

G26. **[30]** If A is the total number of in-person participants at MOAA this year, and B is the total number of online participants at MOAA this year, estimate N, the product AB.

An estimate of e gets a total of $\max(0, 30 - \lceil \log_{10}(8|N - e| + 1) \rceil)$ points.

G27. **[30]** Estimate N, the total number of letters in all the teams that signed up for MOAA this year, both in person and online.

An estimate of e gets a total of $\max\left(0, 30 - \lceil 7\log_5(|N - e| + 1) \rceil\right)$ points.