MOAA 2023 Speed Round Solutions

Math Open at Andover

Ocotber 7th, 2023

S1. Compute $\sqrt{202 \times 3 - 20 \times 23 + 2 \times 23 - 23}$.

Proposed by: Andy Xu

Answer: $\boxed{13}$

Solution: By the order of operations, we do the multiplication first. The expression inside the square root is then $606 - 460 + 46 - 23 = 169$. The answer is then $\sqrt{169} = \boxed{13}$.

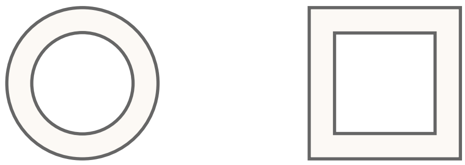
S2. In the coordinate plane, the line passing through points $(2023, 0)$ and $(-2021, 2024)$ also passes through $(1, c)$ for a constant c. Find c.

Proposed by: Andy Xu

Answer: $\boxed{1012}$

Solution: Observe that since $\frac{2023+(-2021)}{2} = 1$, the point $(1, c)$ is the midpoint of $(2023, 0)$ and $(-2021, 2024)$. It follows that $c = \frac{0+2024}{2} = \boxed{1012}$.

S3. Andy and Harry are trying to make an O for the MOAA logo. Andy starts with a circular piece of leather with radius 3 feet and cuts out a circle with radius 2 feet from the middle. Harry starts with a square piece of leather with side length 3 feet and cuts out a square with side length 2 feet from the middle. In square feet, what is the positive difference in area between Andy and Harry's final product to the nearest integer?



Proposed by: Andy Xu

Answer: $\boxed{11}$

Solution: The area of Harry's O is $3^2 - 2^2 = 5$ while the area of Andy's O is $3^2\pi - 2^2\pi = 5\pi$. Thus the positive difference between their areas is $5\pi - 5 = 5(\pi - 1) \approx 5 \cdot 2.14 = 10.7$ so the answer is $\boxed{11}$.

S4. A number is called *super odd* if it is an odd number divisible by the square of an odd prime. For example, 2023 is a *super odd* number because it is odd and divisible by 17^2. Find the sum of all *super odd* numbers from 1 to 100 inclusive.

Proposed by: Andy Xu

Answer: $\boxed{473}$

Solution: Since our range is restricted from 1 to 100 note that the only squares of odd primes that are possible factors of a *super odd* number are 3^2, 5^2, and 7^2 since $11^2 > 100$. In addition, since $3^2 \cdot 5^2 = 15^2 > 100$ any *super odd* number must have exactly one of 3^2, 5^2, or 7^2 as a factor. Since a *super odd* number is also odd, the sum of the *super odd* numbers that are divisible by 3^2 is $3^2(1+3+\cdots+11) = 3^2 \cdot 6^2 = 18^2$. Similarly the sum of the *super odd* numbers that are divisible by 5^2 is $5^2(1+3) = 5^2 \cdot 2^2 = 10^2$ and the sum of the *super odd* numbers that are divisible by 7^2 is just $7^2 \cdot 1 = 7^2$ so our answer is $18^2 + 10^2 + 7^2 = \boxed{473}$.

S5. Let $P(x)$ be a nonzero quadratic polynomial such that $P(1) = P(2) = 0$. Given that $P(3)^2 = P(4) + P(5)$, find $P(6)$.

Proposed by: Andy Xu

Answer: $\boxed{90}$

Solution: Note that if $P(1) = P(2) = 0$ then $P(x) = c(x-1)(x-2)$ for some constant c. Using the fact that $P(3)^2 = P(4) + P(5)$ we have

$$(2c)^2 = 6c + 12c$$

which means

$$4c^2 - 18c = 2c(2c-9) = 0$$

but since $c \neq 0$ we know $2c - 9 = 0$ so $c = \frac{9}{2}$. It follows that $P(6) = \frac{9}{2} \cdot 5 \cdot 4 = \boxed{90}$.

S6. Define the function $f(x) = \lfloor x \rfloor + \lfloor \sqrt{x} \rfloor + \lfloor \sqrt{\sqrt{x}} \rfloor$ for all positive real numbers x. How many integers from 1 to 2023 inclusive are in the range of $f(x)$? Note that $\lfloor x \rfloor$ is known as the *floor* function, which returns the greatest integer less than or equal to x.

Proposed by: Harry Kim

Answer: $\boxed{1973}$

Solution: Every positive real number x which has the same floor function value must produce the same value for $f(x)$ while real numbers with different floor function values must produce different $f(x)$ values. Therefore, we only need to check the greatest integer x where $f(x) \leq 2023$. Observe that $f(1973) = 1973 + 44 + 6 = 2023$ so the answer is $\boxed{1973}$.

S7. Andy flips a strange coin for which the probability of flipping heads is $\frac{1}{2^k+1}$, where k is the number of heads that appeared previously. If Andy flips the coin repeatedly until he gets heads 10 times, what is the expected number of total flips he performs?

Proposed by: Harry Kim

Answer: $\boxed{1033}$

Solution: Observe that the expected number of flips from getting k heads to $k+1$ heads is $2^k + 1$ by linearity of expectation. Therefore, the expected number of total flips can be expressed as $(2^0+1) + (2^1+1) + \cdots + (2^9+1) = 2^{10} - 1 + 10 = \boxed{1033}$.

S8. In the coordinate plane, Yifan the Yak starts at $(0,0)$ and makes 11 moves. In a move, Yifan can either do nothing or move from an arbitrary point (i,j) to $(i+1,j)$, $(i,j+1)$ or $(i+1,j+1)$. How many points (x,y) with integer coordinates exist such that the number of ways Yifan can end on (x,y) is odd?

Proposed by: Yifan Kang

Answer: $\boxed{64}$

Solution: Consider two directions independently. It follows that there are

$$\binom{11}{x}\binom{11}{y}$$

ways to move to point (x, y). Thus, we want to find the number of coordinates such that $\binom{11}{x}$ and $\binom{11}{y}$ are both odd.

We quickly compute or recall some values of $\binom{11}{x}$ (we actually don't need to fully compute each one since we just need the parity, but we've computed them for clarity)

$$\binom{11}{0} = 1, \binom{11}{1} = 11, \binom{11}{2} = 55, \binom{11}{3} = 165, \binom{11}{4} = 330, \binom{11}{5} = 462$$

It follows that there are 8 ways to choose an x such that $\binom{11}{x}$ is odd ($x \in \{0, 1, 2, 3, 8, 9, 10, 11\}$) and similarly 8 ways to choose y, so the answer is $8^2 = \boxed{64}$.

S9. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $BC \perp CD$. There exists a point P on BC such that $\triangle PAD$ is equilateral. If $PB = 20$ and $PC = 23$, the area of $ABCD$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ where b is square-free and a and c are relatively prime. Find $a + b + c$.

Proposed by: Andy Xu

Answer: $\boxed{1854}$

Solution: Let M be the midpoint of AD. The key observation is that $\triangle MBC$ is also equilateral. To prove this we first note that $PM \perp AD$. It follows that $BPMA$ is a cyclic quadrilateral because $\angle PBA + \angle AMP = 90 + 90 = 180$. Thus $\angle PBM = \angle PAM = 60$. Similarly $CPMD$ is cyclic and $\angle PCM = \angle PDM = 60$. Therefore $\triangle MBC$ is equilateral as desired. Let N be the midpoint of BC. Observe that MN is the midline of trapezoid $ABCD$ so

$$[ABCD] = MN \cdot BC = 2 \cdot [MBC] = 2 \cdot \frac{43^2\sqrt{3}}{4} = \frac{1849\sqrt{3}}{2}$$

so our answer is $1849 + 3 + 2 = \boxed{1854}$.

S10. If x, y, z satisfy the system of equations

$$xy + yz + zx = 23$$
$$\frac{y}{x+y} + \frac{z}{y+z} + \frac{x}{z+x} = -1$$
$$\frac{z^2x}{x+y} + \frac{x^2y}{y+z} + \frac{y^2z}{z+x} = 202$$

Find the value of $x^2 + y^2 + z^2$.

Proposed by: Harry Kim

Answer: $\boxed{29}$

Solution: Observe that

$$\begin{aligned}225 &= (xy+yz+zx)+\left(\frac{z^2x}{x+y}+\frac{x^2y}{y+z}+\frac{y^2z}{z+x}\right)\\ &= \left(zx+\frac{z^2x}{x+y}\right)+\left(xy+\frac{x^2y}{y+z}\right)+\left(yz+\frac{y^2z}{z+x}\right)\\ &= \frac{zx(x+y+z)}{x+y}+\frac{xy(x+y+z)}{y+z}+\frac{yz(x+y+z)}{z+x}\\ &= (x+y+z)\left(\frac{zx}{x+y}+\frac{xy}{y+z}+\frac{yz}{z+x}\right)\end{aligned}$$

and

$$\begin{aligned}&(x+y+z)+\left(\frac{zx}{x+y}+\frac{xy}{y+z}+\frac{yz}{z+x}\right)\\ &= \left(x+\frac{zx}{x+y}\right)+\left(y+\frac{xy}{y+z}\right)+\left(z+\frac{yz}{z+x}\right)\\ &= \frac{x(x+y+z)}{x+y}+\frac{y(x+y+z)}{y+z}+\frac{z(x+y+z)}{z+x}\\ &= (x+y+z)\left(\frac{x}{x+y}+\frac{y}{y+z}+\frac{z}{z+x}\right)\\ &= (x+y+z)\left(\left(1-\frac{y}{x+y}\right)+\left(1-\frac{z}{y+z}\right)+\left(1-\frac{x}{z+x}\right)\right)\\ &= 4(x+y+z).\end{aligned}$$

Therefore, we find $\frac{zx}{x+y}+\frac{xy}{y+z}+\frac{yz}{z+x}=3(x+y+z)$ and substituting this into the first equation, we obtain $3(x+y+z)^2=225$ so

$$(x+y+z)^2=75.$$

Hence,

$$x^2+y^2+z^2=(x+y+z)^2-2(xy+yz+zx)=75-46=\boxed{29}.$$