MOAA 2023: Speed Round

October 7th, 2023

Rules

- You have 20 minutes to complete 10 problems. Each answer is a nonnegative integer no greater than 1,000,000.

- If m and n are relatively prime, then the greatest common divisor of m and n is 1.

- No mathematical texts, notes, or online resources of any kind are permitted. Rely on your brain!

- Compasses, protractors, rulers, straightedges, graph paper, blank scratch paper, and writing implements are generally permitted, so long as they are not designed to give an unfair advantage.

- No computational aids (including but not limited to calculators, phones, calculator watches, and computer programs) are permitted on any portion of the MOAA.

- No individual may receive help from any other person, including members of their team. Consulting any other individual is grounds for disqualification.

How to Compete

- **In Person:** After completing the test, write your answers down in the provided Speed Round answer sheet. The proctors will collect your answer sheets immediately after the test ends.

- **Online:** After completing the test, you should input your answers, along with your Team pin and name, into the provided Speed Round Google Form.

Special Thanks to Our Sponsors!

Speed Round Problems

S1. **[2]** Compute $\sqrt{202 \times 3 - 20 \times 23 + 2 \times 23 - 23}$.

S2. **[2]** In the coordinate plane, the line passing through points $(2023, 0)$ and $(-2021, 2024)$ also passes through $(1, c)$ for a constant c. Find c.

S3. **[2]** Andy and Harry are trying to make an O for the MOAA logo. Andy starts with a circular piece of leather with radius 3 feet and cuts out a circle with radius 2 feet from the middle. Harry starts with a square piece of leather with side length 3 feet and cuts out a square with side length 2 feet from the middle. In square feet, what is the positive difference in area between Andy and Harry's final product to the nearest integer?

S4. **[3]** A number is called *super odd* if it is an odd number divisible by the square of an odd prime. For example, 2023 is a *super odd* number because it is odd and divisible by 17^2. Find the sum of all *super odd* numbers from 1 to 100 inclusive.

S5. **[3]** Let $P(x)$ be a nonzero quadratic polynomial such that $P(1) = P(2) = 0$. Given that $P(3)^2 = P(4) + P(5)$, find $P(6)$.

S6. **[4]** Define the function $f(x) = \lfloor x \rfloor + \lfloor \sqrt{x} \rfloor + \lfloor \sqrt{\sqrt{x}} \rfloor$ for all positive real numbers x. How many integers from 1 to 2023 inclusive are in the range of $f(x)$? Note that $\lfloor x \rfloor$ is known as the *floor* function, which returns the greatest integer less than or equal to x.

S7. **[5]** Andy flips a strange coin for which the probability of flipping heads is $\frac{1}{2^k+1}$, where k is the number of heads that appeared previously. If Andy flips the coin repeatedly until he gets heads 10 times, what is the expected number of total flips he performs?

S8. **[6]** In the coordinate plane, Yifan the Yak starts at $(0, 0)$ and makes 11 moves. In a move, Yifan can either do nothing or move from an arbitrary point (i, j) to $(i+1, j)$, $(i, j+1)$ or $(i+1, j+1)$. How many points (x, y) with integer coordinates exist such that the number of ways Yifan can end on (x, y) is odd?

S9. **[6]** Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $BC \perp CD$. There exists a point P on BC such that $\triangle PAD$ is equilateral. If $PB = 20$ and $PC = 23$, the area of $ABCD$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ where b is square-free and a and c are relatively prime. Find $a + b + c$.

S10. **[7]** If x, y, z satisfy the system of equations

$$xy + yz + zx = 23$$

$$\frac{y}{x+y} + \frac{z}{y+z} + \frac{x}{z+x} = -1$$

$$\frac{z^2 x}{x+y} + \frac{x^2 y}{y+z} + \frac{y^2 z}{z+x} = 202$$

Find the value of $x^2 + y^2 + z^2$.