MOAA 2023 Team Round Solutions

Math Open at Andover

October 7th, 2023

T1. Find the last two digits of $2023 + 202^3 + 20^{23}$.

Proposed by: Anthony Yang

Answer: $\boxed{31}$

Solution: Notice that we are just finding $2023 + 202^3 + 20^{23} \pmod{100}$. This is equivalent to

$$2023 + 202^3 + 20^{23} \equiv 23 + 2^3 + 20^{23} \equiv 23 + 8 + 0 \equiv 31 \pmod{100}.$$

so our answer is $\boxed{31}$.

T2. Let $ABCD$ be a square with side length 6. Let E be a point on the perimeter of $ABCD$ such that the area of $\triangle AEB$ is $\frac{1}{6}$ the area of $ABCD$. Find the maximum possible value of CE^2.

Proposed by: Anthony Yang

Answer: $\boxed{52}$

Solution: To maximize CE, we want E to lie on either AB or AD. However, E cannot lie on AB because ΔAEB has positive area, so we want E to lie on AD. Notice that ΔAEB is right, so $[AEB] = \frac{1}{2} \cdot AB \cdot AE$. Further, we already know that $[AEB] = \frac{1}{6} \cdot 6^2 = 6$, so

$$\frac{1}{2} \cdot AB \cdot AE = \frac{1}{2} \cdot 6 \cdot AE = 6$$

which means that $AE = 2$. Thus,

$$CE^2 = DE^2 + CD^2 = (6-2)^2 + 6^2 = \boxed{52}.$$

T3. After the final exam, Mr. Liang asked each of his 17 students to guess the average final exam score. David, a very smart student, received a 100 and guessed the average would be 97. Each of the other 16 students guessed $30 + \frac{n}{2}$ where n was that student's score. If the average of the final exam scores was the same as the average of the guesses, what was the average score on the final exam?

Proposed by: Eric Wang

Answer: $\boxed{62}$

Solution: Let N be the sum of the scores of the other 16 students. Then, the sum of the final exam scores is $100+N$ and the sum of the guesses is $97+30\cdot 16+\frac{N}{2} = 577+\frac{N}{2}$. We get

$$100 + N = 577 + \frac{N}{2} \Rightarrow \frac{N}{2} = 477$$

so $N = 954$. Thus, the average score on the final exam is $\frac{100+N}{17} = \frac{1054}{17} = \boxed{62}$.

T4. Andy has 4 coins c_1, c_2, c_3, c_4 such that the probability that coin c_i with $1 \le i \le 4$ lands tails is $\frac{1}{2^i}$. Andy flips each coin exactly once. The probability that only one coin lands on heads can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Proposed by: Anthony Yang

Answer: $\boxed{525}$

Solution: Note that the probability that coin c_i lands heads is $1 - \frac{1}{2^i} = \frac{2^i-1}{2^i}$. Then, the probability that only coin c_i lands heads after each coin is flipped is $\frac{2^i-1}{2^1 \cdot 2^2 \cdot 2^3 \cdot 2^4} = \frac{2^i-1}{2^{10}}$. Thus, the probability of only one coin landing heads is

$$\sum_{i=1}^{4} \frac{2^i - 1}{2^{10}} = \frac{1+3+7+15}{2^{10}} = \frac{13}{512}$$

so our answer is $13 + 512 = \boxed{525}$.

T5. Angeline starts with a 6-digit number and she moves the last digit to the front. For example, if she originally had 100823 she ends up with 310082. Given that her new number is 4 times her original number, find the smallest possible value of her original number.

Proposed by: Angeline Zhao

Answer: $\boxed{102564}$

Solution: Let x be the desired number and let a be the units digit of x. Then, we have the expression

$$100000a + \frac{x-a}{10} = 4x.$$

Rearranging gives

$$999999a = 39x \Rightarrow 25641a = x.$$

The minimum value of a such that x is a 6-digit number is $a = 4$ so our answer is

$$x = 4 \cdot 25641 = \boxed{102564}.$$

T6. Call a set of integers *unpredictable* if no four elements in the set form an arithmetic sequence. How many unordered *unpredictable* sets of five distinct positive integers $\{a, b, c, d, e\}$ exist such that all elements are strictly less than 12?

Proposed by: Anthony Yang

Answer: $\boxed{367}$

Solution: We will use complementary counting. There are $\binom{11}{5} = 462$ ways to choose the five integers. Now we must subtract all sets that are not *unpredictable*. There are 15 distinct arithmetic sequences $\{a, b, c, d\}$ such that $a, b, c, d < 12$, which leaves $11 - 4 = 7$ integers that e can be. However, we count each arithmetic sequence with 5 elements twice. There are 10 arithmetic sequences $\{a, b, c, d, e\}$ such that $a, b, c, d, e < 12$, so the total number of sets that are not *unpredictable* is $15 \cdot 7 - 10 = 95$. Thus, our answer is $462 - 95 = \boxed{367}$.

T7. In a cube, let M be the midpoint of one of the segments. Choose two vertices of the cube, A and B. What is the number of distinct possible triangles $\triangle AMB$ up to congruency?

Proposed by: Harry Kim

Answer: $\boxed{10}$

Solution: Consider face $\mathcal{F}$ of the cube that contains M. We consider 3 cases:

(i) A and B are both on $\mathcal{F}$

There are 3 possible triangles.

(ii) Only one of A and B are on $\mathcal{F}$

There are 4 possible triangles not congruent to those from the first case

(iii) A and B are not on $\mathcal{F}$

There are 3 possible triangles not congruent to those from the above cases.

Therefore there are $3 + 4 + 3 = \boxed{10}$ possible triangles.

T8. Two consecutive positive integers n and $n + 1$ have the property that they both have 6 divisors but a different number of distinct prime factors. Find the sum of the possible values of n.

Proposed by: Harry Kim

Answer: $\boxed{485}$

Solution: For a positive integer with prime factorization $p_1^{e_1}p_2^{e_2}\cdots p_m^{e_m}$, the number of divisors is $(e_1 + 1)(e_2 + 1)\cdots(e_m + 1)$. Therefore if a positive integer n has 6 divisors, then the prime factorization of n is either p^5 or p^2q. Since a and $a + 1$ have different numbers of prime divisors, there are two cases:

(i) $a = p^5$ and $a + 1 = q^2r$ where p, q, r are distinct prime numbers.

Observe that $p^5+1 = q^2r$. This can be factored into $(p+1)(p^4-p^3+p^2-p+1) = q^2r$. If $p = 2$, then $33 = q^2r$ which is not possible. If $p = 3$, then $4 \cdot 61 = q^2r$ which is possible when $q = 2$ and $r = 61$. If $p \geq 5$, then $p + 1$ is either a power of 2 greater than 4 or a composite number with at least 2 distinct prime divisors. Since $p^4 - p^3 + p^2 - p + 1 > (p + 1)^2$ when $p \geq 5$, this is a contradiction that $(p+1)(p^4 - p^3 + p^2 - p + 1)$ can be expressed as q^2r. Hence the only possible a value is $3^5 = 243$.

(ii) $a = q^2r$ and $a + 1 = p^5$ where p, q, r are distinct prime numbers.

Observe that $p^5-1 = q^2r$. This can be factored into $(p-1)(p^4+p^3+p^2+p+1) = q^2r$. If $p = 2$, then $31 = q^2r$ which is not possible. If $p = 3$, then $2 \cdot 121 = q^2r$ which is possible when $q = 11$ and $r = 2$. We also observe that $p = 5$ is not possible. If $p \geq 7$, then $p-1$ is either a power of 2 greater than 4 or a composite number with at least 2 distinct prime divisors. Since $p^4+p^3+p^2+p+1 > (p-1)^2$ when $p \geq 7$, this is a contradiction that $(p - 1)(p^4 + p^3 + p^2 + p + 1)$ can be expressed as q^2r. Hence the only possible a value is $3^5 - 1 = 243 - 1 = 242$.

Therefore the sum of all possible values of a is $242 + 243 = \boxed{485}$.

T9. Let $ABCDEF$ be an equiangular hexagon. Let P be the point that is a distance of 6 from BC, DE, and FA. If the distances from P to AB, CD, and EF are 8, 11, and 5 respectively, find $(DE - AB)^2$.

Proposed by: Andy Xu

Answer: $\boxed{48}$

Solution: Let AF and BC intersect at X, AF and DE intersect at Y and BC and DE intersect at Z. Let AB and EF intersect at U, let AB and CD intersect at V, and let CD and EF intersect at W. It follows that $\triangle XYZ$ and $\triangle UVW$ are both equilateral. Since P is equidistant from XY, YZ, and ZX, it follows that P is the incenter of $\triangle XYZ$ and the inradius is 6. It can then be quickly computed that $YZ = 12\sqrt{3}$. Now let $UV = s$. It follows that

$$[UVW] = [UVP] + [VWP] + [WUP] = \frac{s \cdot 8}{2} + \frac{s \cdot 11}{2} + \frac{s \cdot 5}{2} = 12s$$

Therefore,

$$12s = \frac{s^2\sqrt{3}}{4}$$

and we get that $s = 16\sqrt{3}$. Finally, observe that $UV - YZ = AB - DE$, so it follows that $(DE - AB)^2 = (UV - YZ)^2 = (4\sqrt{3})^2 = \boxed{48}$.

T10. Let S be the set of lattice points (a, b) in the coordinate plane such that $1 \leq a \leq 30$ and $1 \leq b \leq 30$. What is the maximum number of lattice points in S such that no four points form a square of side length 2?

Proposed by: Harry Kim

Answer: $\boxed{704}$

Solution: Separate the lattice points into 4 groups based on the modulo 2 values of their x and y coordinates. Observe that these groups can be considered separately because the condition that no four points form a square of side length 2 only applies to points in the same group.

Now consider one such group. There are $15 \times 15 = 225$ lattice points. Since no four points can form a 2×2 square, it is easy to see that the maximum number of points that can be chosen for S is $225 - 49 = 176$ (remove all points with odd x value and odd y value). Therefore, the maximum number of points in S is $4 \times 176 = \boxed{704}$.

T11. Let the quadratic $P(x) = x^2 + 5x + 1$. Two distinct real numbers a, b satisfy

$$P(a + b) = ab$$

$$P(ab) = a + b$$

Find the sum of all possible values of a^2.

Proposed by: Harry Kim

Answer: $\boxed{56}$

Solution: Let $p = a + b$ and $q = ab$. Then $f(p) = q$ and $f(q) = p$. Suppose that p and q are distinct. Substituting the values, $f(f(p)) = p$ and $f(f(q)) = q$. Therefore, p and q are two roots of the equation $f(f(x)) = x$. Substituting $f(x) = x^2 + 5x + 1$, we find

$$(x^2 + 5x + 1)^2 + 5(x^2 + 5x + 1) + 1 = x$$
$$x^4 + 10x^3 + 32x^2 + 34x + 7 = 0.$$

Let r and s be the two roots of the equation $f(x) = x$. Then r and s are also the roots of the equation $f(f(x)) = x$ distinct from p and q. By vieta's formula, we obtain

$$p + q + r + s = -10$$

$$pqrs = 7.$$

Notice that r and s are roots of the equation $x^2 + 4x + 1 = 0$, so

$$r + s = -4$$

$$rs = 1.$$

Therefore, we obtain $p + q = -6$ and $pq = 7$. By vieta's formula, p and q are roots of the equation $x^2 + 6x + 7 = 0$. Using the quadratic formula, $(p, q) = (-3 + \sqrt{2}, -3 - \sqrt{2}), (-3 - \sqrt{2}, -3 + \sqrt{2})$. If p and q are not distinct, then either $p = q = r$ or $p = q = s$, so we find $(p, q) = (-2 + \sqrt{3}, -2 + \sqrt{3}), (-2 - \sqrt{3}, -2 - \sqrt{3})$. Notice that $a^2 + b^2 = (a + b)^2 - 2ab = p^2 - 2q$. Since a and b are symmetric, the sum of all possible values of a^2 is the sum of all possible values of $a^2 + b^2 = p^2 - 2q$. Therefore,

$$(-3 + \sqrt{2})^2 - 2(-3 - \sqrt{2}) + (-3 - \sqrt{2})^2 - 2(-3 + \sqrt{2}) = 34$$

$$(-2 + \sqrt{3})^2 - 2(-2 + \sqrt{3}) + (-2 - \sqrt{3})^2 - 2(-2 - \sqrt{3}) = 22$$

Hence the answer is $34 + 22 = \boxed{56}$.

T12. Let N be the number of 105-digit positive integers that contain the digit 1 an odd number of times. Find the remainder when N is divided by 1000.

Proposed by: Harry Kim

Answer: $\boxed{664}$

Solution: The generating function corresponding to this situation is

$$P(x) = (x + 8)(x + 9)^{104},$$

where the degree of x is the number of digit 1. Using roots of unity filter or $\frac{P(1)+P(-1)}{2}$, the number of 105-digit positive integers that contain the digit 1 an even number of times is

$$\frac{1}{2}(9 \cdot 10^{104} + 7 \cdot 8^{104}) \equiv \frac{7 \cdot 8^{104}}{2} \equiv 336 \pmod{1000}$$

Thus, the number of 105-digit positive integers that contain the digit 1 an odd number of times is

$$9 \cdot 10^{104} - 336 \equiv \boxed{664} \pmod{1000}$$

T13. If real numbers x, y, and z satisfy $x^2 - yz = 1$ and $y^2 - xz = 4$ such that $|x + y + z|$ is minimized, then $z^2 - xy$ can be expressed in the form $\sqrt{a} - b$ where a and b are positive integers. Find $a + b$.

Proposed by: Andy Xu

Answer: $\boxed{68}$

Solution: Let $z^2 - xy = c$. Consider the system of equations

$$x^2 - yz = 1$$

$$y^2 - xz = 4$$

$$z^2 - xy = c$$

Subtract pairs of equations to get

$$(y-x)(x+y+z)=3$$

$$(z-y)(x+y+z)=c-4$$

$$(x-z)(x+y+z)=1-c$$

Thus, we know

$$x+y+z=\frac{3}{y-x}=\frac{c-4}{z-y}=\frac{1-c}{x-z}$$

so

$$(x+y+z)^2=\frac{9}{(y-x)^2}=\frac{(c-4)^2}{(z-y)^2}=\frac{(1-c)^2}{(x-z)^2}=\frac{9+(c-4)^2+(1-c)^2}{(y-x)^2+(z-y)^2+(x-z)^2}$$

However, adding the initial equations together reveals that $(y-x)^2+(z-y)^2+(z-x)^2=2(1+4+c)=2(c+5)$ so we have

$$(x+y+z)^2=\frac{9+(c-4)^2+(1-c)^2}{2(c+5)}=\frac{c^2-5c+13}{c+5}=(c-10)+\frac{63}{c+5}$$

It follows that $|x+y+z|=\sqrt{(c-10)+\frac{63}{c+5}}$ so it suffices to minimize the expression $(c-10)+\frac{63}{c+5}$ for $c>-5$ since we require $(c-10)+\frac{63}{c+5}\geq 0$.

If $c>0$ then by AM-GM,

$$(c-10)+\frac{63}{c+5}=c+5+\frac{63}{c+5}-15\geq 2\sqrt{63}-15$$

where equality is achieved when $c=\sqrt{63}-5$. Now consider if $-5<c\leq 0$. Then the function $f(c)=(c-10)+\frac{63}{c+5}$ is decreasing so $f(c)\geq f(0)=2.6>2\sqrt{63}-15$ for $-5<c\leq 0$ so $f(c)$ is indeed minimized when $c=\sqrt{63}-5$ leading to an answer of $\boxed{68}$.

T14. For a positive integer n, let function $f(n)$ denote the number of positive integers $a\leq n$ such that $\gcd(a,n)=\gcd(a+1,n)=1$. Find the sum of all n such that $f(n)=15$.

Proposed by: Harry Kim

Answer: $\boxed{371}$

Solution: For a positive integer $n=p_1^{e_1}\cdots p_k^{e_k}$, we prove that

$$f(n)=n\left(\frac{p_1-2}{p_1}\right)\left(\frac{p_2-2}{p_2}\right)\cdots\left(\frac{p_k-2}{p_k}\right).$$

Let $A_i=\{a\mid a\equiv 0 \pmod{p_i} \text{ or } a+1\equiv 0 \pmod{p_i}\}$. If x is an element of $A_1\cap A_2\cap\cdots\cap A_m$, then

$$\begin{aligned} x &\equiv 0 \text{ or } -1 \pmod{p_1}\\ x &\equiv 0 \text{ or } -1 \pmod{p_2}\\ &\cdots\\ x &\equiv 0 \text{ or } -1 \pmod{p_m}\end{aligned}$$

By the Chinese remainder theorem, there are exactly 2^m number of values of x in $\mod p_1p_2\cdots p_m$ that satisfy this condition. We use the principle of inclusion exclusion to find,

$$\begin{aligned} f(n) &= n - (|A_1| + \cdots + |A_m|) + \sum_{1\le i<j\le k} |A_i \cap A_j| - \cdots + (-1)^k |A_1 \cap \cdots \cap A_k| \\ &= n\left(1 - \left(\frac{2}{p_1} + \frac{2}{p_2} + \cdots + \frac{2}{p_k}\right) + \sum_{1\le i<j\le k} \frac{4}{p_ip_j} - \cdots + (-1)^k \frac{2^k}{p_1p_2\cdots p_k}\right) \\ &= n\left(\frac{p_1-2}{p_1}\right)\left(\frac{p_2-2}{p_2}\right)\cdots\left(\frac{p_k-2}{p_k}\right) \end{aligned}$$

Therefore, $f(n) = p_1^{e_1-1}p_2^{e_2-1}\cdots p_k^{e_k-1}(p_1-2)(p_2-2)\cdots(p_k-2) = 15$. From $\prod_{p|n}(p-2)|3\cdot 5$, the only factors n can have are $3, 5, 7, 17$. If n has 17 as its largest factor, $n = 17, 51$ are possible. If n has 7 as its largest factor, $n = 35, 63, 105$ are possible. If n has 5 as its largest factor, $n = 25, 75$ are possible. Therefore, the answer is $17 + 51 + 35 + 105 + 63 + 25 + 75 = \boxed{371}$.

T15. Triangle ABC has circumcircle ω. Let D be the foot of the altitude from A to BC and let AD intersect ω at $E \neq A$. Let M be the midpoint of AD. If $\angle BMC = 90^\circ$, $AB = 9$ and $AE = 10$, the area of $\triangle ABC$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ where a, b, c are positive integers and b is square-free. Find $a + b + c$.

Proposed by: Andy Xu

Answer: $\boxed{166}$

Solution: The condition of $\angle BMC = 90^\circ$ is equivalent to M lying on the circle with diameter BC. Let line AD intersect this circle at $F \neq M$. By Power of a Point, we see that $BD \cdot CD = AD \cdot DE$ However, by Power of a Point again, we can also write

$$BD \cdot CD = MD \cdot DF = MD^2 = \frac{1}{4}AD^2$$

which implies

$$\frac{1}{4}AD^2 = AD \cdot HD$$

or $AD = 4DE$ immediately giving $AD = 8$. Thus $BD = \sqrt{17}$ and $CD = \frac{AD\cdot DE}{BD} = \frac{16}{\sqrt{17}}$ so

$$[ABC] = \frac{1}{2} \cdot \frac{33}{\sqrt{17}} \cdot 8 = \frac{132\sqrt{17}}{17}$$

giving an answer of $132 + 17 + 17 = \boxed{166}$.