MOAA 2023: Team Round

October 7th, 2023

Rules

- Your team has 40 minutes to complete 15 problems. Each answer is a nonnegative integer no greater than 1,000,000.
- If m and n are relatively prime, then the greatest common divisor of m and n is 1.
- No mathematical texts, notes, or online resources of any kind are permitted. Rely on your brain and those of your teammates!
- Compasses, protractors, rulers, straightedges, graph paper, blank scratch paper, and writing implements are generally permitted, so long as they are not designed to give an unfair advantage.
- No computational aids (including but not limited to calculators, phones, calculator watches, and computer programs) are permitted on any portion of the MOAA.
- Individuals may only receive help from members of their team. Consulting any other individual is grounds for disqualification.

How to Compete

- **In Person:** After completing the test, your team captain should write your answers down in the provided Team Round answer sheet. The proctors will collect your answer sheet immediately after the test ends.
- **Online:** After completing the test, your team captain should input your answers, along with your Team ID and name, into the provided Team Round Google Form.

Team Round Problems

T1. **[5]** Find the last two digits of $2023 + 202^3 + 20^{23}$.

T2. **[10]** Let $ABCD$ be a square with side length 6. Let E be a point on the perimeter of $ABCD$ such that the area of $\triangle AEB$ is $\frac{1}{6}$ the area of $ABCD$. Find the maximum possible value of CE^2.

T3. **[10]** After the final exam, Mr. Liang asked each of his 17 students to guess the average final exam score. David, a very smart student, received a 100 and guessed the average would be 97. Each of the other 16 students guessed $30 + \frac{n}{2}$ where n was that student's score. If the average of the final exam scores was the same as the average of the guesses, what was the average score on the final exam?

T4. **[15]** Andy has 4 coins c_1, c_2, c_3, c_4 such that the probability that coin c_i with $1 \le i \le 4$ lands tails is $\frac{1}{2^i}$. Andy flips each coin exactly once. The probability that only one coin lands on heads can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

T5. **[15]** Angeline starts with a 6-digit number and she moves the last digit to the front. For example, if she originally had 100823 she ends up with 310082. Given that her new number is 4 times her original number, find the smallest possible value of her original number.

T6. **[20]** Call a set of integers *unpredictable* if no four elements in the set form an arithmetic sequence. How many unordered *unpredictable* sets of five distinct positive integers $\{a, b, c, d, e\}$ exist such that all elements are strictly less than 12?

T7. **[20]** In a cube, let M be the midpoint of one of the segments. Choose two vertices of the cube, A and B. What is the number of distinct possible triangles $\triangle AMB$ up to congruency?

T8. **[25]** Two consecutive positive integers n and $n+1$ have the property that they both have 6 divisors but a different number of distinct prime factors. Find the sum of the possible values of n.

T9. **[30]** Let $ABCDEF$ be an equiangular hexagon. Let P be the point that is a distance of 6 from BC, DE, and FA. If the distances from P to AB, CD, and EF are 8, 11, and 5 respectively, find $(DE - AB)^2$.

T10. **[35]** Let S be the set of lattice points (a, b) in the coordinate plane such that $1 \le a \le 30$ and $1 \le b \le 30$. What is the maximum number of lattice points in S such that no four points form a square of side length 2?

T11. **[35]** Let the quadratic $P(x) = x^2 + 5x + 1$. Two distinct real numbers a, b satisfy

$$P(a + b) = ab$$

$$P(ab) = a + b$$

Find the sum of all possible values of a^2.

T12. **[40]** Let N be the number of 105-digit positive integers that contain the digit 1 an odd number of times. Find the remainder when N is divided by 1000.

T13. **[45]** If real numbers x, y, and z satisfy $x^2 - yz = 1$ and $y^2 - xz = 4$ such that $|x + y + z|$ is minimized, then $z^2 - xy$ can be expressed in the form $\sqrt{a} - b$ where a and b are positive integers. Find $a + b$.

T14. [**45**] For a positive integer n, let function $f(n)$ denote the number of positive integers $a \leq n$ such that $\gcd(a, n) = \gcd(a+1, n) = 1$. Find the sum of all n such that $f(n) = 15$.

T15. [**50**] Triangle ABC has circumcircle ω. Let D be the foot of the altitude from A to BC and let AD intersect ω at $E \neq A$. Let M be the midpoint of AD. If $\angle BMC = 90°$, $AB = 9$ and $AE = 10$, the area of $\triangle ABC$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ where a, b, c are positive integers and b is square-free. Find $a + b + c$.