MOAA 2024 Accuracy Round Solutions

Math Open at Andover

October 5, 2024

A1. Compute

$$\frac{2024 \times 2025 - 2021 \times 2022}{2}$$

Proposed by: Anthony Yang

Answer: $\boxed{6069}$

Solution: Recall that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. We have

$$\begin{aligned}\frac{2024 \times 2025 - 2021 \times 2022}{2} &= \frac{2024 \times 2025}{2} - \frac{2021 \times 2022}{2} \\ &= (1 + 2 + \cdots + 2024) - (1 + 2 + \cdots + 2021) \\ &= 2022 + 2023 + 2024 \\ &= \boxed{6069}\end{aligned}$$

A2. Every morning, Anthony walks at a constant speed from his dorm to the Russian School of Math for class. If he walks twice as fast as usual, he gets there 7 minutes earlier. If he walks 1 km/h slower than usual, he gets there 3.5 minutes later. To the nearest integer, how many meters is the Russian School of Math from Anthony's dorm?

Proposed by: Anthony Yang

Answer: $\boxed{1167}$

Solution: First, we convert km/h to meters per minute to get 1 km/h $= \frac{50}{3}$ m/min. Let the usual speed and time be v and t, respectively, and let d be the distance traveled. Then, we have the following system of equations:

$$v \cdot t = d$$

$$2v \cdot (t - 7) = d$$

$$(v - \frac{50}{3}) \cdot (t + \frac{7}{2}) = d$$

Dividing the second equation by the first equation, we get $2 \cdot \frac{t-7}{t} = 1$ which means $t = 14$. Plugging this into the second and third equations gives:

$$14v = d$$

$$\frac{35}{2}(v - \frac{50}{3}) = d$$

Equating the LHS expressions gives $v = \frac{250}{3}$ so

$$d = vt = \frac{250}{3} \cdot 14 = \frac{3500}{3} \approx \boxed{1167}$$

A3. The 9-digit number $\overline{20240MOAA}$ is divisible by the five smallest primes, where M, O, A are (not necessarily distinct) digits. Find the 4-digit number $\overline{MOAA}$.

Proposed by: Anthony Yang

Answer: $\boxed{2200}$

Solution: The five smallest primes are 2, 3, 5, 7, and 11. Notice that our number must be divisible by 10, so $A = 0$. By the divisibility rule of 11, the following expression must be divisible by 11:

$$(2+2+0+O+A)-(0+4+M+A)=(4+O)-(4+M)=O-M$$

Clearly, since O and M are digits, we can only have $O-M=0$ so $O=M$. By the divisibility rule of 3, the sum of the digits must be divisible by 3, or

$$2+0+2+4+0+M+O+A+A=8+2M$$

Checking possibilities, M can only be 2, 5, or 8. Checking $M=2$, we find that 202402200 is divisible by 7, so $\overline{MOAA}=\boxed{2200}$.

A4. A non-decreasing geometric sequence of positive integers with first term a and integer common ratio r is called n-aligned if $a, r \leq n$ and all terms of the sequence yield the same remainder when divided by n. How many 12-aligned sequences exist?

Proposed by: Anthony Yang

Answer: $\boxed{40}$

Solution: Note that if x and y leave the same reminder when divided by a number n, then $x-y$ must be divisible by n. The first two terms of our sequence are a and ar, so we must have

$$ar-a \equiv a(r-1) \equiv 0 \pmod{12}$$

We can proceed with casework on $\gcd(a, 12)$. Notice that if $\gcd(a, 12)=n$, then $\gcd(r-1, 12)$ must be divisible by $\frac{12}{n}$. For example, if $\gcd(a, 12)=1$, then 12 must divide $\gcd(r-1, 12)$ so there are four solutions for a and one solution for r, yielding $4 \cdot 1 = 4$ possibilities. Checking each of the six values for $\gcd(a, 12)$ and summing, we get

$$4 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 + 1 \cdot 6 + 1 \cdot 12 = \boxed{40}$$

A5. Find the number of ordered triples of positive integers (a, b, c) that satisfy

$$abc + 6a + 3b + 2c = 3ab + bc + 2ca + 2024$$

Proposed by: Harry Kim

Answer: $\boxed{11}$

Solution: Rearranging and subtracting 6 from both sides gives

$$abc - 3ab - bc - 2ca + 6a + 3b + 2c - 6 = 2018$$

Notice that this can be factored into

$$(a-1)(b-2)(c-3) = 2018 = 1 \cdot 2 \cdot 1009$$

If $(a-1, b-2, c-3)$ is equivalent to $(1, 2, 1009)$ in some order, there are six possibilities for (a, b, c). If $(a-1, b-2, c-3)$ is equivalent to $(1, 1, 2018)$ in some order, there are three possibilities for (a, b, c). However, notice that we can also have $(a-1, b-2, c-3)$ equivalent to $(2018, -1, -1)$ or $(1009, -1, -2)$, yielding two more solutions. Summing gives $3+6+2=\boxed{11}$.

A6. Two circles ω_1 and ω_2 are defined such that ω_1 is smaller than ω_2. Let X and Y be the intersections of the two circles. Let P be the intersection of their common external tangents, and let W be the intersection of PX and ω_1. The common external tangent closer to X touches ω_1 and ω_2 at A and B respectively, so that $PA = 75$ and $AB = 33$. Given that the quantity WX can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m + n$.

Proposed by: Brandon Xu

Answer: $\boxed{57}$

Solution: Since PA is tangent to ω_1 at A, we have that $\angle PAW = \angle AXW = \angle AXP$. Then, we have that $\triangle PAW \sim \triangle PXA$. Notice that the dilation centered at P which sends A to B, also sends W to X. Then, we have that $AW \parallel BX$, so we have that $\triangle PAW \sim \triangle PBX$ as well. Hence, we have $\triangle PXA \sim \triangle PBX$. By similarity ratios, we have:

$$\frac{PA}{PX} = \frac{PX}{PB} \implies \frac{75}{PX} = \frac{PX}{75+33}$$

Thus, we have $PX = 90$. Notice that by Power of a Point, we have that

$$75^2 = PA^2 = PW \cdot PX = (PX - WX) \cdot PX = 90(90 - WX)$$

Solving for WX gives

$$WX = \frac{90^2 - 75^2}{90} = \frac{55}{2}$$

Hence, our answer is $55 + 2 = \boxed{57}$.

A7. Eric is randomly labeling each cell of a rectangular 8×10 grid with the numbers $1, 2, \ldots, 80$ such that each number is used once. A cell is called a *pit* if its number is smaller than all adjacent cells (two cells are adjacent if they share an edge). If the expected number of *pits* can be expressed as $\frac{a}{b}$ where a and b are relatively prime positive integers, find $a + b$.

Proposed by: Eric Wang

Answer: $\boxed{284}$

Solution: We will compute the expected number of *pits* for each type of cell. A corner cell borders two cells, so by symmetry there is a $\frac{1}{3}$ chance that the corner cell is a *pit*. Similarly, an edge cell borders three cells, meaning there is a $\frac{1}{4}$ chance that the edge cell is a *pit*, and an interior cell borders four cells, meaning there is a $\frac{1}{5}$ chance that the interior cells is a *pit*. There are 4 corner cells, 28 edge cells, and 48 interior cells, so the expected number of pits is

$$4 \cdot \frac{1}{3} + 28 \cdot \frac{1}{4} + 48 \cdot \frac{1}{5} = \frac{269}{15}$$

Thus, our answer is $269 + 15 = \boxed{284}$.

A8. In an equilateral triangle $\triangle ABC$ with area 5, let D be a point on segment AC and E be the point on ray $\overrightarrow{BC}$ not on the segment BC such that $\angle CBD = \angle CDE$ and $BD = 3ED$. The area of $\triangle CDE$ can be expressed as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a + b$.

Proposed by: Harry Kim

Answer: $\boxed{53}$

Solution: Let F be the point on segment BC such that ΔCDF is an equilateral triangle. Notice that $\angle DFB = \angle DCE = 120°$, so we have $\Delta DFB \sim \Delta ECD$. By similarity ratios, we find that

$$\frac{CD}{FB} = \frac{DE}{BD} = \frac{1}{3}$$

Since ΔCDF is equilateral, we have $CD = CF$ so

$$BC = CF + BF = CF + 3CF = 4CF$$

Now, notice that

$$\frac{[\Delta CDF]}{[\Delta ABC]} = \left(\frac{CF}{BC}\right)^2 = \frac{1}{16}$$

where $[\Delta ABC]$ represents the area of ΔABC. Further, observe that since ΔCDE and ΔCDF share the same height, their area ratio is given by

$$\frac{[\Delta CDE]}{[\Delta CDF]} = \frac{CE}{CF}$$

Recall that $CF = FD$ and $\Delta DFB \sim \Delta ECD$. By similarity ratios, we have

$$\frac{CE}{CF} = \frac{CE}{FD} = \frac{CD}{FB} = \frac{1}{3}$$

so $\frac{[\Delta CDE]}{[\Delta CDF]} = \frac{1}{3}$. We find that

$$\frac{[\Delta CDE]}{[\Delta ABC]} = \frac{[\Delta CDE]}{[\Delta CDF]} \cdot \frac{[\Delta CDF]}{[\Delta ABC]} = \frac{1}{3} \cdot \frac{1}{16} = \frac{1}{48}$$

Thus, we get

$$[\Delta CDE] = \frac{1}{48} \cdot [\Delta ABC] = \frac{5}{48}$$

so our answer is $5 + 48 = \boxed{53}$.

A9. Suppose positive real numbers x, y satisfy

$$x\sqrt{2024 - y^2} + y\sqrt{2024 - x^2} = 2024.$$

If the maximum value of $x + 7y$ can be expressed as $a\sqrt{b}$ where a and b are positive integers and b is square-free, find $a + b$.

Proposed by: Harry Kim

Answer: $\boxed{273}$

Solution: From the Cauchy-Schwarz inequality, we have

$$(x^2 + y^2)\left((2024 - y^2) + (2024 - x^2)\right) \geq \left(x\sqrt{2024 - y^2} + y\sqrt{2024 - x^2}\right)^2 = 2024^2$$

Let $A = x^2 + y^2$. We have that

$$A(4048 - A) \geq 2024^2 \implies x^2 + y^2 = 2024$$

We can use Cauchy-Schwarz inequality again to find the maximum of $(x + 7y)$. We have that

$$(x + 7y)^2 \leq (1^2 + 7^2)(x^2 + y^2) = 50 \cdot 2024 \implies x + 7y \leq 20\sqrt{253}$$

so our answer is $20 + 253 = \boxed{273}$.

Remark: this bound is attainable by setting $(x, y) = \left(\sqrt{\frac{2024}{50}}, 7\sqrt{\frac{2024}{50}}\right)$

A10. While doing Math Kangaroo homework, Angeline is painting a 7×7 grid where all of the squares are initially white. For every square on the grid, there is a $\frac{1}{4}$ probability that Angeline paints the square blue, a $\frac{1}{4}$ probability that she paints the square red, a $\frac{1}{4}$ probability that she paints the square purple, and a $\frac{1}{4}$ probability she leaves it white. Let P be the probability that there are more red squares than blue squares. Find the remainder when $2^{100} \cdot P$ is divided by 1000.

Proposed by: Harry Kim

Answer: $\boxed{488}$

Solution: We can convert this problem to having two white 7×7 grids where each square on grid 1 is painted red with $\frac{1}{2}$ probability and each square on grid 2 is painted blue with $\frac{1}{2}$ probability. The two grids can be "combined" so that if a square is colored both red and blue, it turns purple. Then P is equal to the probability that there are more red squares in grid 1 than blue squares in grid 2. Let P' be the probability that there are an equal number of red squares in grid 1 and blue squares in grid 2. Then $P = \frac{1-P'}{2}$ by symmetry. Notice that

$$P' = \sum_{i=0}^{49} \frac{\binom{49}{i}^2}{2^{98}}.$$

Using the identity $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$, we find

$$P' = \frac{\binom{98}{49}}{2^{98}}.$$

Thus,

$$2^{100} \cdot P = 2^{99} - 2 \cdot \binom{98}{49} \equiv \boxed{488} \pmod{1000}.$$