MOAA 2024 Gunga Bowl Solutions

Math Open at Andover

October 5th, 2024

G1. Find the last digit of 2024^{2024}.

Proposed by: Anthony Yang

Answer: $\boxed{6}$

Solution: Notice the last digit of 2024^k is the same as the last digit of 4^k. By simply calculating a few small terms, we can notice a pattern:

$$\begin{aligned} 4^1 &= \underline{\mathbf{4}} \\ 4^2 &= 1\underline{\mathbf{6}} \\ 4^3 &= 6\underline{\mathbf{4}} \\ 4^4 &= 25\underline{\mathbf{6}} \\ 4^5 &= 102\underline{\mathbf{4}} \\ 4^6 &= 409\underline{\mathbf{6}} \\ &\vdots \\ 4^{2024} &= \dots\boxed{6}. \end{aligned}$$

G2. The 2024 MOAA board consists of 4 directors and 8 associates. How many ways can Citadel select a committee of 1 director and 2 associates?

Proposed by: Brandon Xu

Answer: $\boxed{112}$

Solution: There are 4 choices for the director, and $\binom{8}{2}$ ways to choose the two associates, since order does not matter for the choice of associates. Thus, we have

$$4 \cdot \binom{8}{2} = 4 \cdot 28 = \boxed{112}$$

ways of choosing a committee.

G3. While waiting for their Expii class, Cindy and David test how well they can count. Cindy starts by counting 1, 5, 9, and so on, adding 4 each time. David starts by counting 400, 397, 394, and so on, subtracting 3 each time. If they start counting at the same time and count at the same rate, what number will Cindy and David say at the same time?

Proposed by: Brandon Xu

Answer: $\boxed{229}$

Solution: Suppose they say one number each second (i.e. Cindy says 1 on the first second, 5 on the second second, and so on, and similarly for David). Suppose Cindy and David say the same number at the k^{th} second. Then, at the k^{th} second, we can see that Cindy will say the number $4k-3$, while David will say the number $403-3k$. Setting these equal to each other, we have

$$4k-3 = 403-3k \implies 7k = 406 \implies k = 58$$

Substituting in 58 for k, we get $4 \cdot 58 - 3 = 403 - 3 \cdot 58 = \boxed{229}$.

G4. Let $ABCD$ be a square with side length 4. Let E be a point such that ΔACE is an equilateral triangle. If S is the area of ΔACE, find S^2.

Proposed by: Brandon Xu

Answer: $\boxed{192}$

Solution: We can use the Pythagorean Theorem on $\triangle ABC$ to find that $AC^2 = 4^2 + 4^2 = 32$. Plugging this into the formula for the area of an equilateral triangle gives

$$S = \frac{AC^2\sqrt{3}}{4} = 8\sqrt{3} \implies S^2 = \boxed{192}$$

G5. How many factors of 2024×2025 have an odd number of divisors?

Proposed by: Brandon Xu

Answer: $\boxed{12}$

Solution: Recall the fact that a number has an odd number of divisors if and only if it is a perfect square. Let $N = 2024 \times 2025$. We can factor $N = 2^3 \cdot 3^4 \cdot 5^2 \cdot 11 \cdot 23$. Any perfect square factor of N must take the form $2^a 3^b 5^c 11^d 23^e$ where a, b, c, d, e are even integers (possibly zero). Clearly, $d = e = 0$. We can see that a can be either 0 or 2, b can be 0, 2, or 4, and c can be 0 or 2. Hence, the number of total perfect square factors is just $2 \cdot 3 \cdot 2 = \boxed{12}$.

G6. A four-digit number is called *heavy* if the sum of its first three digits is equal to its units digit. For example, 2024 is a *heavy* number because $2+0+2 = 4$. How many *heavy* numbers are there?

Proposed by: Harry Kim

Answer: $\boxed{165}$

Solution: We will express a four-digit number as $\overline{abcd}$, so that $a+b+c = d$, and $a > 0$. We can rewrite this equation as $a' + b + c = d - 1$, where $a' = a - 1$. Then, we just have the condition that $a', b, c \geq 0$. By stars and bars, the number of ordered triples (a', b, c) which satisfy this equation is just $\binom{d+1}{2}$. Hence, the number of *heavy* numbers is just

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{10}{2}$$

By the Hockey Stick Identity, or by simple enumeration, this sum evaluates to $\binom{11}{3} = \boxed{165}$.

G7. Let $ABCD$ be a quadrilateral such that $AB = 3$, $BC = 6$, $CD = 4$, $DA = 5$, and $\angle ABC + \angle ADC = 180°$. If lines DA and BC meet at point E, find the perimeter of ΔABE.

Proposed by: Anthony Yang

Answer: $\boxed{36}$

Solution: Note that $\angle ABE = 180^\circ - \angle ABC = \angle ADC$. Since $\angle ABE = \angle ADC$ and $\angle BEA = \angle DEC$, we have $\Delta ABE \sim \Delta CDE$. By similarity ratios, we can rewrite as

$$\frac{BE}{DE} = \frac{AE}{CE} = \frac{AB}{CD} = \frac{3}{4}$$

Notice that $CE = BE + 6$ and $DE = AE + 5$, so we have

$$\frac{BE}{AE+5} = \frac{AE}{BE+6} = \frac{3}{4}$$

We get the following system of equations:

$$4BE = 3AE + 15$$

$$3BE + 18 = 4AE$$

Summing the two together gives

$$4BE + 4AE = 3AE + 3BE + 33 \implies AE + BE + 33$$

Thus, the perimeter of ΔABE is $AB + BE + AE = 3 + 33 = \boxed{36}$.

G8. Let x, y be numbers such that $x + y = 3$ and

$$\left(\frac{x}{y}\right)^y + \left(\frac{y}{x}\right)^x = \frac{1}{x^x y^y}$$

If the value of xy can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers, find $a + b$.

Proposed by: Anthony Yang

Answer: $\boxed{35}$

Solution: Multiplying both sides by $x^x y^y$ gives

$$x^{x+y} + y^{x+y} = 1$$

Since $x + y = 3$, this can be rewritten as $x^3 + y^3 = 1$. Factoring the LHS gives

$$\begin{aligned} x^3 + y^3 &= (x+y)(x^2 - xy + y^2) \\ &= (x+y)\left((x+y)^2 - 3xy\right) \\ &= (3)(3^2 - 3xy) \\ &= 27 - 9xy \end{aligned}$$

We now have $27 - 9xy = 1$, and solving gives $xy = \frac{26}{9}$ so our answer is $26 + 9 = \boxed{35}$.

G9. The 2024 MOAA board consists of 4 directors and 8 associates. They want to watch a movie at the theaters for a team celebration. If there are N ways for the 12 team members to sit in a row such that no three consecutive members are all directors or all associates, find $\frac{N}{8!}$.

Proposed by: Brandon Xu

Answer: $\boxed{360}$

Solution: Let M be the number of ways to permute 4 D's and 8 A's so that no three consecutive letters are the same. In particular, note that $N = M \cdot 4! \cdot 8!$, since the directors can be freely permuted among the spots they take, and similarly for the associates. Number the seats from left to right with the integers from 1 to 12. Let a_i be the number of A's to the left of the i-th D for $1 \le i \le 4$ (i.e. a_1 is the number of A's to the left of the first D) and let a_5 be the number of A's to the right of the last D. In particular, note that $a_1 + a_2 + a_3 + a_4 + a_5 = 8$. For the seating arrangement to be valid, all a_i must be less than 3, and no two consecutive a_i among a_2, a_3, a_4 can be zero. However, the latter condition is not necessary, as it is implied by the former. From here, we can simply list possibilities:

a) $(a_1, a_2, a_3, a_4, a_5) = (2, 2, 2, 2, 0)$
There are 5 ways in this case, by choosing which a_i is 0.

b) $(a_1, a_2, a_3, a_4, a_5) = (2, 2, 2, 1, 1)$
There are 10 ways in this case, by choosing which two a_i are equal to 1.

There are no other possibilities, so we have that $M = 5 + 10 = 15$. We are looking for $\frac{N}{8!} = M \cdot 4! = 15 \cdot 24 = \boxed{360}$.

G10. Angela has an unfair six-sided dice such that each odd number has probability p of being rolled while each even number has probability q of being rolled. After rolling the dice twice, the probability that the product of her rolls is a perfect square is $\frac{3}{4}(p+q)$. If pq can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers, find $a + b$.

Proposed by: Anthony Yang

Answer: $\boxed{49}$

Solution: Since there are three odd numbers and three even numbers, we have

$$3p + 3q = 1 \implies p + q = \frac{1}{3}$$

Thus, the probability that the product of her rolls is a perfect square is $\frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$. We now wish to express this probability differently in terms of p and q. There are two ways for the product to be a perfect square: Angela rolls the same number twice, or one of the rolls is a 1 and the other is a 4. The first case happens with probability $3p^2 + 3q^2$, while the second case happens with probability $2pq$ since the 1 could be rolled first or second. We now have

$$3p^2 + 3q^2 + 2pq = \frac{1}{4}$$

We can rewrite the LHS as

$$\begin{aligned} 3p^2 + 3q^2 + 2pq &= 3(p+q)^2 - 4pq \\ &= 3(\frac{1}{3})^2 - 4pq \\ &= \frac{1}{3} - 4pq \end{aligned}$$

so we have $\frac{1}{3} - 4pq = \frac{1}{4}$. Solving gives $pq = \frac{1}{48}$, so our answer is $1 + 48 = \boxed{49}$.

G11. Let $ABCDEF$ be a regular hexagon with side length 6. Let G and H be points such that $ACGH$ is a square containing point E. If AG intersects DE at point X, the length of AX can be expressed as $\sqrt{a}-\sqrt{b}$, where a and b are positive integers. Find $a+b$.

Proposed by: Anthony Yang

Answer: $\boxed{864}$

Solution:

G12. Define a recursive sequence by $a_1 = \frac{9}{10}$, $a_2 = \frac{1}{8}$, and

$$a_{n+1} = \frac{a_n a_{n-1}}{a_n + a_{n-1}}$$

for all $n \geq 2$. Find the positive integer k such that $a_k = \frac{1}{2024}$.

Proposed by: Anthony Yang

Answer: $\boxed{14}$

Solution: Define a second sequence by $b_n = \frac{1}{a_n}$. We have $b_1 = \frac{10}{9}$, $b_2 = 8$, and

$$b_{n+1} = \frac{1}{a_{n+1}} = \frac{a_n + a_{n-1}}{a_n a_{n-1}} = \frac{1}{a_{n-1}} + \frac{1}{a_n} = b_{n-1} + b_n$$

Let F_n denote the n^{th} term of the Fibonacci sequence, where $F_0 = F_1 = 1$. Notice that we can write

$$b_n = F_{n-3}b_1 + F_{n-2}b_2 = \frac{10}{9}F_{n-3} + 8F_{n-2}$$

Thus, we will always have $\frac{10}{9}(F_{n-3} + F_{n-2}) < b_n < 8(F_{n-3} + F_{n-2})$, which can be re-expressed as

$$\frac{10}{9}F_{n-1} < b_n < 8F_{n-1}$$

We wish to find the n such that $b_n = 2024$, or similarly the n such that

$$\frac{10}{9}F_{n-1} < 2024 < 8F_{n-1}$$

This can be rewritten as the following inequality:

$$253 < F_{n-1} < \frac{9108}{5}$$

Further, since $8F_{n-2}$ is an integer, $\frac{10}{9}F_{n-3}$ must also be an integer, so 9 must divide F_{n-3}. Checking the Fibonacci numbers reveals that $n = 14$ is the only solution, so our answer is $\boxed{14}$.

G13. The intersections of the graphs of $6x = 25y^2 - 1$ and $5y = 36x^2 - 1$ form a convex quadrilateral with diagonals intersecting at point P. Given that the coordinates of P can be written as (x, y), find $\frac{1}{xy}$.

Proposed by: Brandon Xu

Answer: $\boxed{120}$

Solution: As the intersections must satisfy both equations, we can subtract the two equations to get

$$6x - 5y = 25y^2 - 36x^2$$

Rearranging gives $25y^2 - 36x^2 + 5y - 6x = 0$, which can be factored into

$$(5y - 6x)(5y + 6x + 1) = 0$$

Thus, either $5y - 6x = 0$ or $5y + 6x + 1 = 0$. After graphing, we notice that these lines are the diagonals of the quadrilateral. Thus, the intersection of these lines is the same as the intersection of the diagonals. Solving these equations gives $(x, y) = \left(-\frac{1}{12}, -\frac{1}{10}\right)$, so our answer is $\boxed{120}$.

G14. Let ΔABC have sides $AB = 3$, $BC = 5$, and $CA = 7$. Circle ω_1 passes through point B and is tangent to AC at point A, and circle ω_2 passes through B and is tangent to AC at point C. Let O_1 and O_2 be the centers of ω_1 and ω_2 respectively. If the area of quadrilateral AO_1O_2C can be expressed as $\frac{a}{b\sqrt{c}}$ where a and b are relatively prime positive integers and c is square-free, find $a + b + c$.

Proposed by: Brandon Xu

Answer: $\boxed{851}$

Solution: Applying Heron's formula on $\triangle ABC$, we get that

$$[ABC] = \sqrt{\left(\frac{15}{2}\right)\left(\frac{15}{2} - 3\right)\left(\frac{15}{2} - 5\right)\left(\frac{15}{2} - 7\right)} = \frac{15\sqrt{3}}{4}$$

Let D be the foot of the altitude from B to AC, E be the foot of the altitude from O_1 to AB, and F be the foot of the altitude from O_2 to BC. Notice that $\angle O_1AB = 2\angle BAC$ and $\angle O_2BC = 2\angle BCA$, since AC is tangent to ω_1 and ω_2. Then, we have that $\triangle O_1EA \sim \triangle ADB$ and $\triangle O_2FC \sim \triangle CDB$. Similarity ratios gives us

$$\frac{O_1A}{AB} = \frac{AE}{BD} \implies \frac{O_1A}{3} = \frac{\frac{3}{2}}{\frac{15\sqrt{3}}{14}}$$

and

$$\frac{O_2C}{BC} = \frac{CF}{BD} \implies \frac{O_2C}{5} = \frac{\frac{5}{2}}{\frac{15\sqrt{3}}{14}}$$

Thus, $O_1A = \frac{7\sqrt{3}}{5}$ and $O_2C = \frac{35\sqrt{3}}{9}$. As AO_1O_2C is a trapezoid with bases AO_1 and O_2C and height AC, we have

$$[AO_1O_2C] = \frac{7}{2} \cdot \left(\frac{7\sqrt{3}}{5} + \frac{35\sqrt{3}}{9}\right) = \frac{833}{15\sqrt{3}}$$

so our answer is $833 + 15 + 3 = \boxed{851}$.

G15. Let M be a two-digit integer $\overline{ab}$ and let N be a three-digit integer $\overline{cde}$ that satisfies

$$3MN = \overline{abcde}.$$

Find the sum of all possible values of $\overline{abcde}$.

Proposed by: Harry Kim

Answer: $\boxed{127011}$

Solution: Rewrite $\overline{abcde} = 1000M + N$. Notice that for $1000M + N$ to be divisible by M, we must have $\frac{1000M+N}{M} = 1000 + \frac{N}{M}$ be an integer, which directly implies that N is divisible by M. Now, let $N = kM$ for some positive integer k. We have

$$3MN = 1000M + N \implies 3kM^2 = (1000 + k)M$$

Dividing both sides by M gives $3kM = 1000+k$. Thus, we want to find all values of k such that $M = \frac{1000+k}{3k}$ is a two-digit integer and $kM < 1000$. Note that the latter inequality directly implies that $k < 100$. First, in order for $1000 + k$ to be divisible by k, we must have that 1000 is divisible by k. Further, $1000 + k$ must be divisible by 3 meaning $k \equiv 2 \pmod 3$. The possible values of k are then 2, 5, 8, 20, and 50. These cases yield $M = 167$, 67, 42, 17, and 7, respectively. 167 and 7 are clearly not two-digit numbers, so we have three solutions. Since $N = kM$, our three numbers are 67335, 42336, and 17340, so our answer is $67335 + 42336 + 17340 = \boxed{127011}$.

G16. Harry has a strange calculator from Husdon River Trading that has two buttons. Button A multiplies a number by 2 and adds 1 while button B multiplies a number by 5 and adds 4. For example, if the number 5 is on the screen, pressing A would turn the number into 11 and pressing B would turn the number into 29. If the initial number is 1, find the sum of all possible numbers less than 100 that the calculator can produce.

Proposed by: Harry Kim

Answer: $\boxed{414}$

Solution: Let N be the current number on the screen. Let $M = N + 1$. Since we begin with $N = 1$, it follows that M initially equals $1 + 1 = 2$. Notice that Button A sends M to $2M$ and Button B sends M to $5M$. We can now enumerate all possible values of M less than or equal to 100 by repeatedly multiplying by 2 or 5. All such values of M can be written in the form $2^{a+1}5^b$, where a and b are nonnegative integers. We can simply list all numbers of this form, by doing casework on b:

a) **Case 1:** $b = 0$. We have $M \in \{2, 4, 8, 16, 32, 64\}$.
b) **Case 2:** $b = 1$. We have $M \in \{10, 20, 40, 80\}$.
c) **Case 2:** $b = 2$. We have $M \in \{50, 100\}$.

Summing these elements, and subtracting 1 from each number gives

$$(2 + 4 + 8 + 16 + 32 + 64) + (10 + 20 + 40 + 80) + (50 + 100) - 12 = \boxed{414}.$$

G17. Points A, B, C, D lie on sphere S_1 such that $AB = CD = 5$, $AC = BD = 7$, and $BC = AD = 8$. Sphere S_2 is the largest sphere contained inside tetrahedron $ABCD$. Given that the product of the radii of S_1 and S_2 can be expressed as $\frac{a\sqrt{b}}{c}$ where a and c are relatively prime positive integers and b is square-free, find $a + b + c$.

Proposed by: Brandon Xu

Answer: $\boxed{258}$

Solution: Let the radius of S_1 be r_1, and the radius of S_2 be r_2. Notice that tetrahedron $ABCD$ can be inscribed inside a rectangular prism so that the pairs of equal sides on $ABCD$ correspond to face diagonals on opposite faces of the rectangular prism. Then, we can see that the center of this rectangular prism is equidistant to A,B,C, and D so we have

$$r_1 = \frac{1}{2} \cdot \sqrt{a^2 + b^2 + c^2}$$

where a, b, and c are the sides of this prism. We can set up equations to get:

$$a^2+b^2=5^2$$
$$b^2+c^2=7^2$$
$$a^2+c^2=8^2$$

Solving gives $a=2\sqrt{5}$, $b=\sqrt{5}$, and $c=2\sqrt{11}$, giving $r_1=\frac{\sqrt{69}}{2}$. To find r_2, we will calculate the volume V of $ABCD$ in two ways. First, we can see that

$$V=\frac{abc}{3}=\frac{20\sqrt{11}}{3}$$

by cutting off 4 corners of the prism, each with volume equal to $\frac{1}{6}$ of the total volume of the prism. However, we also have

$$\begin{aligned} V &= r_2\cdot\frac{[ABC]+[ABD]+[BCD]+[ACD]}{3} \\ &= r_2\cdot\frac{4[ABC]}{3} \\ &= r_2\cdot\frac{40\sqrt{3}}{3} \end{aligned}$$

Solving for r_2 gives $r_2=\frac{\sqrt{33}}{6}$. Finally, we have $r_1r_2=\frac{\sqrt{253}}{4}$ so our answer is $a+b+c=1+253+4=\boxed{258}$.

G18. Positive integers a,b,c satisfy the equation

$$a+\frac{b}{a}+\frac{c}{b}=2024$$

Suppose that b and c are relatively prime. Find the maximum value of $a+b$.

Proposed by: Harry Kim

Answer: $\boxed{1980}$

Solution: We must have that $\frac{b}{a}+\frac{c}{b}$ is an integer. Since b and c are relatively prime, $\frac{c}{b}$ is already in its most reduced form, meaning we must have $\frac{b}{a}=\frac{n}{b}$ where n is an integer such that n is relatively prime to b. However, notice that if $\frac{b}{a}=\frac{n}{b}$, we must have

$$b^2=na \implies a=\frac{b^2}{n}$$

Since a is an integer, this means that $\gcd(b,n)\neq 1$ whenever $n\neq 1$. Thus, we must have $n=1$, so $a=b^2$. Substituting this in gives

$$a+\frac{b}{a}+\frac{c}{b}=b^2+\frac{c+1}{b}=2024$$

Note that we must have $b^2<2024$, since c is a positive integer. Since $a+b$ increases as b increases, we wish to maximize b. The largest possible value of b such that $b^2<2024$ is 44, so $a+b=b^2+b=1936+44=\boxed{1980}$.

G19. From positive integers $1,2,\ldots,2024$, two (not necessarily distinct) numbers a and b are picked uniformly at random. The expected value of $\dfrac{a^2}{b^2+b}$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

Proposed by: Harry Kim

Answer: $\boxed{4055}$

Solution: The expected value of $\frac{a^2}{b^2+b}$ is given by:

$$\frac{1}{2024^2}\sum_{a=1}^{2024}\sum_{b=1}^{2024}\frac{a^2}{b^2+b}$$

Notice that for a fixed b, the factor $\frac{1}{b^2+b}$ remains constant with respect to a. Thus, we can separate the two summations to obtain:

$$\frac{1}{2024^2}\sum_{a=1}^{2024}\sum_{b=1}^{2024}\frac{a^2}{b^2+b} = \frac{1}{2024}\sum_{a=1}^{2024}a^2 \cdot \frac{1}{2024}\sum_{b=1}^{2024}\frac{1}{b^2+b}$$

In other words, we can find the expected value of a^2 and the expected value of $\frac{1}{b^2+b}$ and multiply the two. We know that $1+2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$, so the expected value of a^2 is just

$$\frac{1}{2024}\sum_{a=1}^{2024}a^2 = \frac{1}{2024}\frac{(2024)(2024+1)(2024\cdot 2+1)}{6} = \frac{675\cdot 4049}{2}$$

The expected value of $\frac{1}{b^2+b}$ is given by

$$\frac{1}{2024}\sum_{b=1}^{2024}\frac{1}{b^2+b} = \frac{1}{2024}\sum_{b=1}^{2024}\frac{1}{b(b+1)} = \frac{1}{2024}\sum_{b=1}^{2024}\left(\frac{1}{b}-\frac{1}{b+1}\right)$$

Notice that this is a telescoping sum, so we have

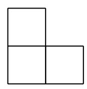
$$\frac{1}{2024}\sum_{b=1}^{2024}\left(\frac{1}{b}-\frac{1}{b+1}\right) = \frac{1}{2024}\left(\frac{1}{1}-\frac{1}{2025}\right) = \frac{1}{2025}$$

Thus, the expected value of $\frac{a^2}{b^2+b}$ is

$$\frac{1}{2024}\sum_{a=1}^{2024}a^2 \cdot \frac{1}{2024}\sum_{b=1}^{2024}\frac{1}{b^2+b} = \frac{675\cdot 4049}{2}\cdot\frac{1}{2025} = \frac{4049}{6}$$

so our answer is $4049+6 = \boxed{4055}$.

G20. How many ways are there to tile a 4×15 grid using rotations of the following shape?



Proposed by: Harry Kim

Answer: $\boxed{2672}$

Solution:

G21. Find the number of positive integers n less than 2024 such that $x^6+x^4+x^3+x^2+1$ divides

$$x^{10n}+x^{9n}+x^{8n}+x^{7n}+x^{6n}-5$$

Proposed by: Harry Kim

Answer: $\boxed{67}$

Solution: First observe that $x^6+x^4+x^3+x^2+1=(x^2-x+1)(x^4+x^3+x^2+x+1)$. Let ω be a root of $x^2-x+1=0$. Notice that $\omega^3+1=0$. For the condition to hold, $\omega^{10n}+\omega^{9n}+\omega^{8n}+\omega^{7n}+\omega^{6n}=5$. This is equivalent to

$$\omega^{4n}+\omega^{3n}+\omega^{2n}+\omega^n+1=5$$

since $\omega^{6n}=1$. If n is not a multiple of 3, it is easy to check that the condition is not satisfied (as the resulting value would be complex). If n is a multiple of 3 but not a multiple of 6 then the resulting value is 1 which is a contradiction. Hence n is a multiple of 6 and this works. Similarly, let α be a root of $x^4+x^3+x^2+x+1=0$. Then it is easy to see that n has to be a multiple of 5 since all other cases would result in 0 when α is put in. Therefore, n is a multiple of 30. Thus the answer is $\frac{2023}{30}=\boxed{67}$.

G22. Six cards numbered 1 through 6 are stacked so that the cards are in an ascending order from top to bottom (1 is at top, 6 is at bottom). Nate shuffles the deck using a method called the top shuffle. A top shuffle is an operation where the card at the top of the stack is moved to a randomly chosen position that is not at the top. The relative order of all other cards remain unchanged. Once the card numbered 1 goes to the bottom, Nate considers the deck well-shuffled and stops shuffling. The expected number of times Nate performs the top shuffle can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

Proposed by: Harry Kim

Answer: $\boxed{235}$

Solution: Let E_i be the expected number of top shuffles Nate performs when the card number 1 is in the ith position. Then we obtain the following system of equations:

$$\begin{aligned}
E_1 &= 1+\frac{1}{5}(E_2+E_3+E_4+E_5)\\
E_2 &= E_1+1\\
E_3 &= E_2+\frac{5}{4}\\
E_4 &= E_3+\frac{5}{3}\\
E_5 &= E_4+\frac{5}{2}
\end{aligned}$$

Solving this system of equations, we find $E_1=\frac{223}{12}$. Thus, the answer is $223+12=\boxed{235}$.

G23. Let $ABCD$ be a rectangle such that $AB>AD$ and $AD=6$. Let P lie on AB such that $2AP=PB$ and Q be a point in the interior of $ABCD$ such that $AD=AQ$ and $\angle PQB=45^\circ$. Find the area of $ABCD$.

Proposed by: Anthony Yang

Answer: $\boxed{108}$

Solution: After the competition, we noticed that the problem statement was invalid. Specifically, the configuration is not unique given the conditions in the problem. We sincerely apologize for this mistake. Fortunately, there is no change to the rankings and results.

G24. Real numbers x, y, z satisfy the system of equations

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} = \frac{y}{x} + \frac{z}{y} + \frac{x}{z}$$

$$\frac{x^3}{y} + \frac{2y^2}{z} + \frac{4z}{y^2} + \frac{4y}{x^3} = 59$$

The maximum possible value of $x+y+z$ can be expressed in the form $\frac{a+b\sqrt{c}}{2}$, where a, b, c are positive integers, and c is square-free. Find $a + b + c$.

Proposed by: Harry Kim

Answer: $\boxed{65}$

Solution: Let $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$ so that $abc = 1$. Then notice that by the rearrangement inequality,

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} = a^2 + b^2 + c^2 \geq ab + bc + ca = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{y}{x} + \frac{z}{y} + \frac{x}{z}.$$

The equality case is $a = b = c = 1$, so we obtain $x = y = z$. Now we can transform the second equation to

$$x^2 + 2x + \frac{4}{x} + \frac{4}{x^2} = 59.$$

Let $t = x + \frac{2}{x}$. Then the equation again transforms to

$$t^2 + 2t = 63.$$

Then $(t + 9)(t - 7) = 0$ so $t = 7, -9$. When $t = 7$, we have $x^2 - 7x + 2 = 0$ so $x = \frac{7 \pm \sqrt{41}}{2}$ using the quadratic formula.

When $t = -9$, we have $x^2 + 9x + 2 = 0$ so $x = \frac{-9 \pm \sqrt{73}}{2}$ using the quadratic formula. Thus, the maximum value of $x + y + z = 3x$ is $\frac{21+3\sqrt{41}}{2}$. Hence, the answer is $21 + 3 + 41 = \boxed{65}$.