MOAA 2024: Gunga Bowl

October 5th, 2024

Gunga Bowl Problems

Gunga Bowl Set 1

G1. **[8]** Find the last digit of 2024^{2024}.

G2. **[8]** The 2024 MOAA board consists of 4 directors and 8 associates. How many ways can Citadel select a committee of 1 director and 2 associates?

G3. **[8]** While waiting for their Expii class, Cindy and David test how well they can count. Cindy starts by counting 1, 5, 9, and so on, adding 4 each time. David starts by counting 400, 397, 394, and so on, subtracting 3 each time. If they start counting at the same time and count at the same rate, what number will Cindy and David say at the same time?

Gunga Bowl Set 2

G4. **[10]** Let $ABCD$ be a square with side length 4. Let E be a point such that ΔACE is an equilateral triangle. If S is the area of ΔACE, find S^2.

G5. **[10]** How many factors of 2024×2025 have an odd number of divisors?

G6. **[10]** A four-digit number is called *heavy* if the sum of its first three digits is equal to its units digit. For example, 2024 is a *heavy* number because $2 + 0 + 2 = 4$. How many *heavy* numbers are there?

Gunga Bowl Set 3

G7. **[12]** Let $ABCD$ be a quadrilateral such that $AB = 3$, $BC = 6$, $CD = 4$, $DA = 5$, and $\angle ABC + \angle ADC = 180°$. If lines DA and BC meet at point E, find the perimeter of ΔABE.

G8. **[12]** Let x, y be numbers such that $x + y = 3$ and

$$\left(\frac{x}{y}\right)^y + \left(\frac{y}{x}\right)^x = \frac{1}{x^x y^y}$$

If the value of xy can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers, find $a + b$.

G9. **[12]** The 2024 MOAA board consists of 4 directors and 8 associates. They want to watch a movie at the theatres for a team celebration. If there are N ways for the 12 team members to sit in a row such that no three consecutive members are all directors or all associates, find $\frac{N}{8!}$.

Gunga Bowl Set 4

G10. [**14**] Angela has an unfair six-sided dice such that each odd number has probability p of being rolled while each even number has probability q of being rolled. After rolling the dice twice, the probability that the product of her rolls is a perfect square is $\frac{3}{4}(p+q)$. If pq can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers, find $a+b$.

G11. [**14**] Let $ABCDEF$ be a regular hexagon with side length 6. Let G and H be points such that $ACGH$ is a square containing point E. If AG intersects DE at point X, the length of AX can be expressed as $\sqrt{a}-\sqrt{b}$, where a and b are positive integers. Find $a+b$.

G12. [**14**] Define a recursive sequence by $a_1 = \frac{9}{10}$, $a_2 = \frac{1}{8}$, and

$$a_{n+1} = \frac{a_n a_{n-1}}{a_n + a_{n-1}}$$

for all $n \geq 2$. Find the positive integer k such that $a_k = \frac{1}{2024}$.

Gunga Bowl Set 5

G13. [**16**] The intersections of the graphs of $6x = 25y^2 - 1$ and $5y = 36x^2 - 1$ form a convex quadrilateral with diagonals intersecting at point P. Given that the coordinates of P can be written as (x, y), find $\frac{1}{xy}$.

G14. [**16**] Let ΔABC have sides $AB = 3$, $BC = 5$, and $CA = 7$. Circle ω_1 passes through point B and is tangent to AC at point A, and circle ω_2 passes through B and is tangent to AC at point C. Let O_1 and O_2 be the centers of ω_1 and ω_2 respectively. If the area of quadrilateral AO_1O_2C can be expressed as $\frac{a}{b\sqrt{c}}$ where a and b are relatively prime positive integers and c is square-free, find $a+b+c$.

G15. [**16**] Let M be a two-digit integer $\overline{ab}$ and let N be a three-digit integer $\overline{cde}$ that satisfies

$$3MN = \overline{abcde}.$$

Find the sum of all possible values of $\overline{abcde}$.

Gunga Bowl Set 6

G16. [**18**] Harry has a strange calculator from Husdon River Trading that has two buttons. Button A multiplies a number by 2 and adds 1 while button B multiplies a number by 5 and adds 4. For example, if the number 5 is on the screen, pressing A would turn the number into 11 and pressing B would turn the number into 29. If the initial number is 1, find the sum of all possible numbers less than 100 that the calculator can produce.

G17. [**18**] Points A, B, C, D lie on sphere S_1 such that $AB = CD = 5$, $AC = BD = 7$, and $BC = AD = 8$. Sphere S_2 is the largest sphere contained inside tetrahedron $ABCD$. Given that the product of the radii of S_1 and S_2 can be expressed as $\frac{a\sqrt{b}}{c}$ where a and c are relatively prime positive integers and b is square-free, find $a+b+c$.

G18. [**18**] Positive integers a, b, c satisfy the equation

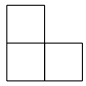
$$a + \frac{b}{a} + \frac{c}{b} = 2024$$

Suppose that b and c are relatively prime. Find the maximum value of $a+b$.

Gunga Bowl Set 7

G19. **[20]** From positive integers $1, 2, \ldots, 2024$, two (not necessarily distinct) numbers a and b are picked uniformly at random. The expected value of $\dfrac{a^2}{b^2+b}$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

G20. **[20]** How many ways are there to tile a 4×15 grid using rotations of the following shape?



G21. **[20]** Find the number of positive integers n less than 2024 such that $x^6 + x^4 + x^3 + x^2 + 1$ divides

$$x^{10n} + x^{9n} + x^{8n} + x^{7n} + x^{6n} - 5$$

Gunga Bowl Set 8

G22. **[22]** Six cards numbered 1 through 6 are stacked so that the cards are in an ascending order from top to bottom (1 is at top, 6 is at bottom). Nate shuffles the deck using a method called the top shuffle. A top shuffle is an operation where the card at the top of the stack is moved to a randomly chosen position that is not at the top. The relative order of all other cards remain unchanged. Once the card numbered 1 goes to the bottom, Nate considers the deck well-shuffled and stops shuffling. The expected number of times Nate performs the top shuffle can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

G23. **[22]** Let $ABCD$ be a rectangle such that $AB > AD$ and $AD = 6$. Let P lie on AB such that $2AP = PB$ and Q be a point in the interior of $ABCD$ such that $AD = AQ$ and $\angle PQB = 45^\circ$. Find the area of $ABCD$.

G24. **[22]** Real numbers x, y, z satisfy the system of equations

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} = \frac{y}{x} + \frac{z}{y} + \frac{x}{z}$$
$$\frac{x^3}{y} + \frac{2y^2}{z} + \frac{4z}{y^2} + \frac{4y}{x^3} = 59$$

The maximum possible value of $x+y+z$ can be expressed in the form $\frac{a+b\sqrt{c}}{2}$, where a, b, c are positive integers, and c is square-free. Find $a+b+c$.

Gunga Bowl Set 9

This set consists of three estimation problems, with scoring schemes described.

G25. **[30]** Estimate N, the total number of participants (in person and online) at MOAA this year.

An estimate of e gets a total of $\max\left(0, \left\lfloor 150\left(1 - \frac{|N-e|}{N}\right)\right\rfloor - 120\right)$ points.

G26. **[30]** Estimate L, the total number of letters in all the teams that signed up for MOAA this year, both in person and online.

An estimate of e gets a total of $\max\left(0, 30 - \lceil 7\log_5(|L-e|+1)\rceil\right)$ points.

G27. **[30]** Estimate S, the sum of the correct answers to every question across every round of MOAA 2024 (besides Gunga Bowl Set 9).

An estimate of e gets a total of $\max(0, 30 - \lceil \log_{10}(8|S-e|+1)\rceil)$ points.