# MOAA 2024: Team Round

October 5th, 2024

## Rules

- Your team has 40 minutes to complete 15 problems. Each answer is a nonnegative integer no greater than 1,000,000.
- If $m$ and $n$ are relatively prime, then the greatest common divisor of $m$ and $n$ is 1.
- No mathematical texts, notes, or online resources of any kind are permitted. Rely on your brain and those of your teammates!
- Compasses, protractors, rulers, straightedges, graph paper, blank scratch paper, and writing implements are generally permitted, so long as they are not designed to give an unfair advantage.
- No computational aids (including but not limited to calculators, phones, calculator watches, and computer programs) are permitted on any portion of the MOAA.
- Individuals may only receive help from members of their team. Consulting any other individual is grounds for disqualification.

## How to Compete

- **In Person:** After completing the test, your team captain should write your answers down in the provided Team Round answer sheet. The proctors will collect your answer sheet immediately after the test ends.
- **Online:** Log into the Classtime session to access the test. Input all answers directly into the provided form. Select for the test to be handed in once you are ready.

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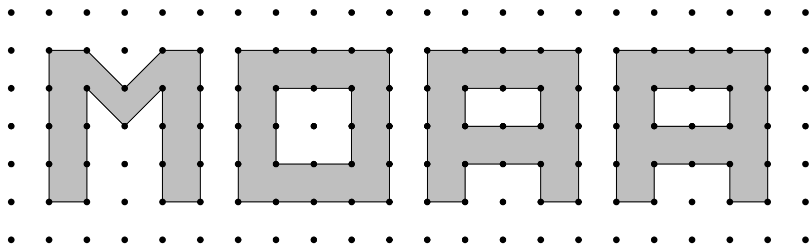
## Team Round Problems

T1. **[5]** Compute
$$\frac{20\times 21\times 22+21\times 22\times 23+22\times 23\times 24}{20+21+22+23+24}$$

T2. **[10]** Angeline is making an MOAA sign out of paper for her Areteem homework and plans on using the design below. If adjacent points are 1 cm apart and she only uses exactly enough paper to fill in the shaded regions, how much paper will she need in $\text{cm}^2$?



T3. **[10]** Brandon is buying letters at Jane Street, where the cost of each letter is equivalent to its position in the alphabet in dollars. For example, the letter $A$ costs \$1 and the letter $M$ costs \$13. Suppose the cost of a word is the product of the costs of each of its letters (here, a word is defined as any string of letters). How many words with distinct letters cost \$64?

T4. **[15]** Valencia has six magical coins purchased from Expii that either land heads, land tails, or disappear with equal probability. When a coin disappears, it is gone forever. After Valencia flips all six magical coins simultaneously, the probability that more coins land heads than tails can be expressed as $\frac{a}{b}$, where $a$ and $b$ are relatively prime positive integers. Find $a+b$.

T5. **[15]** A positive integer is *exclusive* if it has distinct, odd digits and is divisible by each of its digits. For example, 15 is *exclusive* because it has distinct, odd digits and it is divisible by 1 and 5. Find the greatest *exclusive* integer.

T6. **[20]** A cubic polynomial $P(x)$ with integer coefficients satisfies the following three equations:

a) $P(-1)\cdot P(1)=5$

b) $P(-1)\cdot P(-2)=20$

c) $P(1)\cdot P(2)=32$

Given that $P(1)>0$, find the value of $P(3)$.

T7. **[20]** David is collecting coins in the coordinate plane. At any point $(x,y)$, he collects a coin with probability $\frac{x}{x+y}$ and moves to either $(x+1,y)$ or $(x,y+1)$. If he takes any route from $(1,1)$ to $(2024,2024)$ with equal probability, the expected number of coins he will collect on his journey can be expressed as $\frac{a}{b}$, where $a$ and $b$ are relatively prime positive integers. Find $a+b$. (Note: he can collect coins at both his start and end points)

T8. **[25]** Points $A$, $B$, and $C$ lie on line $\ell$ in that order such that $AB=24$ and $BC=31$. Let $D$ and $E$ be points lying on the same side of $\ell$ such that $\Delta ABD$ and $\Delta BCE$ are equilateral triangles. Let $F$ be the point closer to $\ell$ such that $\Delta DEF$ is an equilateral triangle. Find the value of $AF+BF+CF$.

T9. **[30]** Let $f(k)$ denote the $k^{\text{th}}$ smallest positive integer that is not a perfect square. For example, $f(1) = 2$, $f(2) = 3$, and $f(3) = 5$. Let $n$ be the positive integer that satisfies

$$f^{2024}(n) = 2024^2 + 1$$

where $f^{2024}$ denotes the function $f$ composed 2024 times. Find the remainder when $n$ is divided by 1000.

T10. **[35]** Anika and Angela are playing a game of tic-tac-toe. They take turns drawing shapes in a $3 \times 3$ grid. Anika goes first and draws an X each turn and Angela draws an O each turn. The first player who gets three of their shapes in a line (vertical, horizontal, diagonal) wins the game. Suppose that Anika won the game. Compute the remainder of the number of possible final configurations of the $3 \times 3$ grid divided by 1000.

| | | |
|---|---|---|
| X | O | |
| X | | O |
| X | O | X |

T11. **[35]** Let $\overline{abcdef}$ be the unique six-digit number with non-zero digits such that

$$(\overline{abc} - \overline{def})^2 = \overline{abcdef}$$

Find the value of $\overline{abcdef}$.

T12. **[40]** For a positive integer $n$, let $f_k(n)$ denote the number of positive divisors of $n$ less than or equal to $k$. Suppose two (not necessarily distinct) positive integers $a, b$ are randomly chosen from the set $\{1, 2, \cdots 1200\}$. If the expected value of $f_6(ab)$ can be expressed as $\frac{a}{b}$ where $a$ and $b$ are relatively prime positive integers, find $a + b$.

T13. **[45]** A ladybug is in the center of a cube with side length 4, such that it is distance 2 away from every face of the cube. The ladybug wants to touch four faces of the cube and return to its original position (the ladybug "touches" a face of the cube if it is at any interior or boundary point of the face). If $M$ is the minimum distance the ladybug needs to travel, find $M^2$.

T14. **[45]** Cindy and Zadie are playing a game. There are three piles of candies, $A, B, C$, each with $a, b, c$ candies respectively where $1 \leq a, b, c \leq 8$. Cindy begins the game and each turn, a player chooses a pile and eats any positive number of candies in that pile. The player who eats the last remaining candy loses. What is the number of triples $(a, b, c)$ such that Cindy can guarantee a win no matter how Zadie plays?

T15. **[50]** Two circles $\omega_1$ and $\omega_2$ with radii 16 and 3, respectively, are internally tangent at point $A$. Let $B$ and $C$ be points on $\omega_1$ such that $BC$ is tangent to $\omega_2$ and $\angle BAC = 120°$. The value of $AB + AC$ can be expressed as $\frac{a\sqrt{b}}{c}$ where $a$ and $c$ are relatively prime positive integers and $b$ is square-free. Find $a + b + c$.