MOAA 2024 Speed Round Solutions

Math Open at Andover

October 5th, 2024

S1. Compute $2024 + 202 \times 4 + 20 \times 24 + 2 \times 24$.

Proposed by: Anthony Yang

Answer: $\boxed{3360}$

Solution: By order of operations, we multiply first then add:

$$2024 + 808 + 480 + 48 = \boxed{3360}$$

S2. Let $ABCD$ be a rectangle with $AB = 12$ and $AD = 6$. Let E be a point on segment AB such that the area of ΔAED is 24. Find the perimeter of $BCDE$.

Proposed by: Anthony Yang

Answer: $\boxed{32}$

Solution: Since $\angle EAD = 90^\circ$, we know the area of $\triangle AED$ is just equal to $\frac{1}{2} \cdot AE \cdot AD$. Solving for AE gives $AE = 8$. By the Pythagorean Theorem on $\triangle AED$, we have $ED = \sqrt{6^2 + 8^2} = 10$. Then, the perimeter of $BCDE$ is just

$$(12 - 8) + 6 + 12 + 10 = \boxed{32}$$

S3. The integers $1, 2, 3, \ldots, 19, 20$ are written in a row on a blackboard at AlphaStar Academy. Nate draws a vertical line between two consecutive numbers n and $n + 1$. Given that the sum of the numbers on the left of Nate's line and the sum of the numbers on the right of his line are equal, find n.

Proposed by: Anthony Yang and Brandon Xu

Answer: $\boxed{14}$

Solution: The sum of the numbers on the left of Nate's line is just:

$$1 + 2 + \cdots + n$$

Since this sum is equal to the sum of the numbers on the right of Nate's line, this sum must be equal to half of the total sum of the integers from 1 to 20. We have

$$2 \cdot (1 + 2 + \cdots + n) = 1 + 2 + \cdots + 20$$

Recall that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, so we have

$$2 \cdot \frac{n(n + 1)}{2} = \frac{20 \cdot 21}{2} \implies n(n + 1) = 210$$

Solving the final equation gives $n = \boxed{14}$.

S4. Harry has a fair 4-sided dice while Brandon has a fair 8-sided dice. After they both roll their dice once, the probability that Brandon's number is larger than Harry's number can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find the value of $a+b$.

Proposed by: Anthony Yang

Answer: $\boxed{27}$

Solution:

Notice that if Brandon rolls a number greater than 4, he is guaranteed to have a larger number than Harry. This happens with probability $\frac{1}{2}$. Otherwise, both Brandon and Harry roll a number in the range of 1 to 4, which happens with probability $\frac{1}{2}$. There are three possibilities: either Brandon rolls the same number as Harry, Brandon rolls the smaller number, or Brandon rolls the larger number. Notice the latter two cases have the same probability by symmetry. The first case happens with probability $\frac{1}{4}$. Then, Brandon rolls a larger number than Harry with probability $\frac{1-\frac{1}{4}}{2} = \frac{3}{8}$. Thus, our desired probability is $\frac{1}{2} + \frac{1}{2} \cdot \frac{3}{8} = \boxed{\frac{11}{16}}$.

S5. Let x and y be two positive integers that satisfy the following equations:

$$x + y = 33$$
$$\gcd(x, y) + \operatorname{lcm}(x, y) = 87$$

Find xy.

Proposed by: Brandon Xu and Harry Kim

Answer: $\boxed{252}$

Solution: Let $x' = \frac{x}{\gcd(x,y)}$ and $y' = \frac{y}{\gcd(x,y)}$. Further, note that

$$\operatorname{lcm}(x, y) \cdot \gcd(x, y) = xy \implies \operatorname{lcm}(x, y) = \frac{xy}{\gcd(x, y)} = \gcd(x, y)x'y'$$

Thus, we get the following system of equations:

$$\gcd(x, y) \cdot (x' + y') = 33$$
$$\gcd(x, y) \cdot (1 + x'y') = 87$$

Notice that $\gcd(x, y)$ must divide $\gcd(33, 87) = 3$, so $\gcd(x, y) = 1, 3$. If $\gcd(x, y) = 1$, then we have

$$x' + y' = 33$$
$$1 + x'y' = 87 \implies x'y' = 86$$

Since $86 = 43 \cdot 2$, we can easily check and find that there are no solutions. If $\gcd(x, y) = 3$, then we have

$$x' + y' = 11$$
$$1 + x'y' = 29 \implies x'y' = 28$$

We find that $x' = 4$ and $y' = 7$, so $x = 12$ and $y = 21$. Our answer is $12 \cdot 21 = \boxed{252}$.

S6. Let ΔABC be a triangle with $AB = 12$. The angle bisector of $\angle ABC$ intersects side AC at point D such that $AD = 8$ and $BD = CD$. Find the length of BC.

Proposed by: Anthony Yang

Answer: $\boxed{15}$

Solution: Since BD is the angle bisector of $\angle ABC$, we have $\angle ABD = \angle CBD$. Further, since ΔBDC is isosceles, we have $\angle CBD = \angle BCD$. Let $\angle BCD = \alpha$. Then, we have $\angle BDC = 180^\circ - 2\alpha$ which means

$$\angle ADB = 180^\circ - \angle BDC = 2\alpha$$

Now note that $\angle ABC = 2\alpha$. We have $\angle ABC = \angle ADB$ and $\angle BAD = \angle CAB$ which implies that $\Delta ADB \sim \Delta ABC$. By similarity ratios, we have

$$\frac{AB}{AC} = \frac{AD}{AB} = \frac{2}{3}$$

so $AC = \frac{3}{2} \cdot AB = 18$. Thus, we get $BD = DC = AC - AD = 18 - 8 = 10$. By similarity ratios again, we have

$$\frac{BC}{BD} = \frac{AB}{AD} = \frac{3}{2}$$

so $BC = \frac{3}{2} \cdot BD = \boxed{15}$.

S7. A set S contains 13 distinct positive integers, and satisfies the following two conditions:

a) The number of distinct sums $a + b$ where a, b are two (not necessarily distinct) elements of S is 25.

b) The median of S is 38.

Find the smallest possible value for an element of S.

Proposed by: Brandon Xu

Answer: $\boxed{2}$

Solution: We claim that the elements of S must form an arithmetic sequence. Suppose that the elements of S are $a_1, a_2, \cdots a_{13}$,so that $a_1 < a_2 < \cdots < a_{13}$. Consider a sequence of points forming a path on the lattice grid connecting $(1, 1)$ and $(13, 13)$ moving either up or right. Moreover, label each lattice point (i, j) with the sum $a_i + a_j$. Note that each "step" either up or right must end on a point with a larger label than the point before. Further, note that there must be 25 total "steps", meaning that such a path will exactly cover all 25 sums. For every $1 \le i, j < 13$, there exists a path whose $(i + j - 1)^{\text{st}}$ step lands on $(i + 1, j)$ and a path whose $(i + j - 1)^{\text{st}}$ step lands on $(i, j + 1)$. Then, we must have

$$a_{i+1} + a_j = a_i + a_{j+1}$$

as otherwise, there would be more than 25 sums. We can rewrite this equation to

$$a_{i+1} - a_i = a_{j+1} - a_j$$

Thus, the elements of S must form an arithmetic sequence. We have that $a_7 = 38$. Letting the common difference of the arithmetic sequence be d, we have that $a_1 = 38 - 6d > 0$, so $d \le 6$. When $d = 6$, we get the minimum value of a_1 to be $38 - 36 = \boxed{2}$.

S8. Let x, y, z be positive real numbers satisfying

$$\frac{16 - x^2 + 2yz}{(y+z)^2} + \frac{16 - y^2 + 2xz}{(x+z)^2} + \frac{16 - z^2 + 2xy}{(x+y)^2} = 3$$

The minimum possible value of xyz can be expressed as $\frac{a\sqrt{b}}{c}$, where a, b, c are positive integers such that a and c are relatively prime and b is square-free. Find $a + b + c$.

Proposed by: Brandon Xu

Answer: $\boxed{76}$

Solution: Note that

$$\begin{aligned} 16 - x^2 - y^2 - z^2 &= 16 - x^2 + 2yz - (y+z)^2 \\ &= 16 - y^2 + 2xz - (x+z)^2 \\ &= 16 - z^2 + 2xy - (x+y)^2 \end{aligned}$$

This motivates us to subtract 3 from both sides:

$$\frac{16 - x^2 + 2yz}{(y+z)^2} + \frac{16 - y^2 + 2xz}{(x+z)^2} + \frac{16 - z^2 + 2xy}{(x+y)^2} - 3 = 0$$

Notice that this can be written as

$$\left(\frac{16 - x^2 + 2yz}{(y+z)^2} - 1\right) + \left(\frac{16 - y^2 + 2xz}{(x+z)^2} - 1\right) + \left(\frac{16 - z^2 + 2xy}{(x+y)^2} - 1\right) = 0$$

which then gives

$$\frac{16 - x^2 - y^2 - z^2}{(y+z)^2} + \frac{16 - x^2 - y^2 - z^2}{(x+z)^2} + \frac{16 - x^2 - y^2 - z^2}{(x+y)^2} = 0$$

Factoring finally gives

$$(16 - x^2 - y^2 - z^2)\left(\frac{1}{(y+z)^2} + \frac{1}{(x+z)^2} + \frac{1}{(x+y)^2}\right) = 0$$

Note that the second term in this expansion is always positive, so we must have that $16 - x^2 - y^2 - z^2 = 0$. By AM-GM, we find that xyz is maximized when $x^2 = y^2 = z^2 = \frac{16}{3}$. Then,

$$xyz = \left(\frac{4}{\sqrt{3}}\right)^3 = \frac{64\sqrt{3}}{9}$$

so our answer is $64 + 3 + 9 = \boxed{76}$.

S9. For a set S, an element $a \in S$ is called an *atom* if there **does not** exist distinct elements $b, c \in S$ such that $a = b + c$. Let $U = \{1, 2, \ldots, 2024\}$. A subset A of U is chosen uniformly at random from all of the subsets of U (including the empty set and the full set). Suppose N is the expected number of *atoms* in A. What is the closest integer to N?

Proposed by: Harry Kim

Answer: $\boxed{4}$

Solution: By linearity of expectation, the expected number of atoms in A is equal to the sum of probabilities that each of $1, 2, \ldots, 2024$ is an atom. For each element, there is a $\frac{1}{2}$ probability that they are included in A. It is easy to see that 1 and 2 must be an atom if they are included in A. For integers $n \geq 3$, there are $\lfloor \frac{n-1}{2} \rfloor$ combinations of $x, y \in U$ such that $n = x + y$. There is a $\frac{1}{4}$ probabilities that both $x, y \in A$, so there is a $\frac{3}{4}$ probability that this pairing does not exist. Hence,

$$N = \frac{1}{2}\Big(1 + 1 + \frac{3}{4} + \frac{3}{4} + \frac{9}{16} + \frac{9}{16} + \ldots\Big) = \sum_{i=0}^{1011} \left(\frac{3}{4}\right)^i \simeq \frac{1}{1 - \frac{3}{4}} = \boxed{4}.$$

S10. Let $P(x)$ be a polynomial with degree at least 1 that satisfies

$$P(x^2 + 1) - P(x^2 - 1) = kxP(x) + 18x^2$$

for all real numbers x where k is a constant. Find the value of $P(k)$.

Proposed by: Harry Kim

Answer: $\boxed{630}$

Solution: Substituting $x \to -x$, we obtain

$$P(x^2 + 1) - P(x^2 - 1) = -kxP(-x) + 18x^2 = kxP(x) + 18x^2.$$

Thus, $-P(-x) = P(x)$ and $P(x)$ is an odd function. Since $P(x)$ is a polynomial, $P(x)$ only contains terms with odd degrees. We also see that $P(x)$ has degree at least 3 since the degree of the left hand side would be a constant if $P(x)$ had degree 1. Now, let the degree of $P(x)$ be n and the leading coefficient be $a \neq 0$. Comparing the degree of the left hand side and the right hand side, $2n - 2 = n + 1$. Thus, $n = 3$. Comparing the leading coefficients, $2an = ka$. Thus, $k = 6$.

Let $P(x) = ax^3 + bx$. Substituting $x = 1$, we obtain $P(2) - P(0) = 6P(1) + 18$. This gives $8a + 2b = 6a + 6b + 18$ so

$$a - 2b = 9.$$

Substituting $x = 2$, we obtain $P(5) - P(3) = 12P(2) + 72$ so

$$a - 11b = 36$$

Solving the linear equations, we obtain $a = 3$ and $b = -3$. Therefore,

$$P(k) = P(6) = 3 \cdot 6^3 - 3 \cdot 6 = \boxed{630}.$$