MOAA 2024 Team Round Solutions

Math Open at Andover

October 5th, 2024

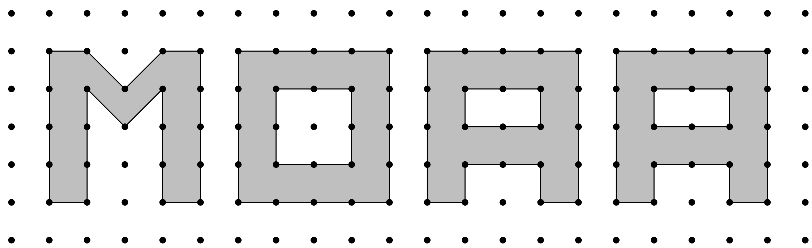
T1. Compute
$$\frac{20 \times 21 \times 22 + 21 \times 22 \times 23 + 22 \times 23 \times 24}{20 + 21 + 22 + 23 + 24}$$

Proposed by: Anthony Yang

Answer: $\boxed{291}$

Solution: Notice that the denominator can be rewritten as $22 \cdot 5$, so we have
$$\frac{20 \times 21 \times 22 + 21 \times 22 \times 23 + 22 \times 23 \times 24}{20 + 21 + 22 + 23 + 24} = \frac{20 \cdot 21 + 21 \cdot 23 + 23 \cdot 24}{5}$$
By order of operations on the numerator, we multiply first then add, which gives $\frac{1455}{5} = \boxed{291}$.

T2. Angeline is making an MOAA sign out of paper for her Areteem homework and plans on using the design below. If adjacent points are 1 cm apart and she only uses exactly enough paper to fill in the shaded regions, how much paper will she need in cm^2?



Proposed by: Anthony Yang

Answer: $\boxed{46}$

Solution: Each unit square has area 1 cm^2, and the two parallelograms in the M each also have area 1 cm^2. Simply counting in each letter, we find 44 unit squares and two parallelograms, so our answer is $44 + 2 = \boxed{46}$.

T3. Brandon is buying letters at Jane Street, where the cost of each letter is equivalent to its position in the alphabet in dollars. For example, the letter A costs \$1 and the letter M costs \$13. Suppose the cost of a word is the product of the costs of each of its letters (here, a word is defined as any string of letters). How many words with distinct letters cost \$64?

Proposed by: Brandon Xu

Answer: $\boxed{38}$

Solution: We want to find the number of ways to factor 64 into distinct factors, where the order of the factors matters. We can simply list the possibilities:

$$\begin{aligned} 4 \cdot 16 \\ 1 \cdot 4 \cdot 16 \\ 2 \cdot 4 \cdot 8 \\ 1 \cdot 2 \cdot 4 \cdot 8 \end{aligned}$$

Since the number of words using m letters is just $m!$, we have $2! + 3! + 3! + 4! = \boxed{38}$ total words satisfying the condition.

T4. Valencia has six magical coins purchased from Expii that either land heads, land tails, or disappear with equal probability. When a coin disappears, it is gone forever. After Valencia flips all six magical coins simultaneously, the probability that more coins land heads than tails can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

Proposed by: Anthony Yang

Answer: $\boxed{341}$

Solution: By symmetry, the probability that more coins land heads than tails is equivalent to the probability that more coins land tails than heads. It suffices to find the probability that an equal number of coins land heads and tails. There are four cases: no coins disappear, two coins disappear, four coins disappear, and six coins disappear.

a) **Case 1:** No coins disappear.

   There $\binom{6}{3} = 20$ ways to pick the coins that land heads, so there are 20 possible flippings for this case.

b) **Case 2:** Two coins disappear.

   There are $\binom{6}{2}$ ways to pick the coins that disappear, and $\binom{4}{2}$ ways to pick the coins that land heads. There are $\binom{6}{2} \cdot \binom{4}{2} = 90$ possible flippings for this case.

c) **Case 3:** Four coins disappear.

   There are $\binom{6}{4}$ ways to pick the coins that disappear, and $\binom{2}{1}$ ways to pick the coin that lands heads. There are $\binom{6}{4} \cdot \binom{2}{1} = 30$ possible flippings for this case.

d) **Case 4:** Six coins disappear.

   There is $\binom{6}{6} = 1$ way to pick the coins that disappear, so there is 1 possible flipping for this case.

Summing these up yields 141 ways to have an equal number of coins land heads and tails. Since there are $3^6 = 729$ total flippings, the probability that an equal number of coins land heads and tails is $\frac{141}{729} = \frac{47}{243}$. Thus, the probability that more coins land heads than tails is

$$\frac{1 - \frac{47}{243}}{2} = \frac{98}{243}$$

so our answer is $98 + 243 = \boxed{341}$.

T5. A positive integer is *exclusive* if it has distinct, odd digits and is divisible by each of its digits. For example, 15 is *exclusive* because it has distinct, odd digits and it is divisible by 1 and 5. Find the greatest *exclusive* integer.

Proposed by: Jialai She and Anthony Yang

Answer: $\boxed{9315}$

Solution: Since there are five odd digits, the greatest *exclusive* number has at most five digits. However, any five-digit *exclusive* number has a digit sum of $1+3+5+7+9=25$, meaning the number would not be divisible by 3. Thus, there are no five-digit *exclusive* numbers, so the greatest *exclusive* number has four digits.

We can now decide which digit to not include. In order to obtain the largest *exclusive* number, we want the first digit to be 9. Then, the number must be divisible by 9, meaning its digit sum must also be divisible by 9. This is only possible when 7 is removed. Thus, our four digits are 1, 3, 5, and 9. Clearly, the units digit must be 5, and we already set the first digit to be 9. This leaves two ways to arrange the 3 and the 1. Since both arrangements give *exclusive* numbers, we simply choose the arrangement which yields the larger number. Thus, the largest *exclusive* number is $\boxed{9315}$.

T6. A cubic polynomial $P(x)$ with integer coefficients satisfies the following three equations:

a) $P(-1) \cdot P(1) = 5$

b) $P(-1) \cdot P(-2) = 20$

c) $P(1) \cdot P(2) = 32$

Given that $P(1) > 0$, find the value of $P(3)$.

Proposed by: Anthony Yang

Answer: $\boxed{109}$

Solution: Notice that if $P(x)$ has integer coefficients, then $P(x)$ must be an integer as long as x is an integer. Thus, from the first equation, we know that $P(1) = 1$ or 5. However, from the third equation, we know that $P(1) \neq 5$ or else $P(2)$ would not be an integer, so we must have $P(1) = 1$ and $P(-1) = 5$. Substituting these values into the second and third equations gives $P(-2) = 4$ and $P(2) = 32$. Now let

$$P(x) = ax^3 + bx^2 + cx + d$$

for integer coefficients a, b, c, d. Then, we have the following system of equations:

$$\begin{aligned}
P(-1) &= -a + b - c + d = 5 \\
P(1) &= a + b + c + d = 1 \\
P(-2) &= -8a + 4b - 2c + d = 4 \\
P(2) &= 8a + 4b + 2c + d = 32
\end{aligned}$$

Adding the first two equations gives $2b + 2d = 6 \implies b + d = 3$, and subtracting this from the second equation gives $a + c = -2$. Now let $d = 3 - b$ and $c = -2 - a$. Plugging these into the third and fourth equation gives the following:

$$\begin{aligned}
-6a + 3b + 7 = 4 &\implies -2a + b = -1 \\
6a + 3b - 1 = 32 &\implies 2a + b = 11
\end{aligned}$$

Solving gives $b = 5$ and $a = 3$, so $d = -2$ and $c = -5$. Thus, we find that $P(x) = 3x^3 + 5x^2 - 5x - 2$ so

$$P(3) = 3(3^3) + 5(3^2) - 5(3) - 2 = \boxed{109}$$

T7. David is collecting coins in the coordinate plane. At any point (x, y), he collects a coin with probability $\frac{x}{x+y}$ and moves to either $(x+1, y)$ or $(x, y+1)$. If he takes any route from $(1, 1)$ to $(2024, 2024)$ with equal probability, the expected number of coins he will collect on his journey can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$. (Note: he can collect coins at both his start and end points)

Proposed by: Anthony Yang

Answer: $\boxed{4049}$

Solution 1: First, note that David must make exactly 2023 horizontal and 2023 vertical moves in order to end at $(2024, 2024)$. Every path therefore has 4046 moves, and hence visits $4046 + 1 = 4047$ points. Now fix any path P from $(1, 1)$ to $(2024, 2024)$. Define its reflection P' as the path obtained by replacing every horizontal move in P with a vertical move, and vice versa. For example, if H represents a horizontal move and V represents a vertical move, the reflection of the path $HHVV$ would be $VVHH$. After k steps, suppose path P is at the point (h_k, v_k). The key observation is that after k steps, its reflection P' will be at the point (v_k, h_k). Notice that the expected total number of coins collected at this point in path P and P' is

$$\frac{h_k}{h_k + v_k} + \frac{v_k}{h_k + v_k} = 1$$

Thus, for *each* pair of corresponding points in P and P', the expected total number of coins collected is 1. Since David visits 4047 lattice points, the expected total number of coins collected across path P and P' is

$$1 \cdot 4047 = 4047$$

By symmetry (and since David takes any path with equal probability), the expected number of coins collected in P is equivalent to P'. Thus, the expected number of coins David collects in a single path P is just $\frac{4047}{2}$, so our answer is $4047+2 = \boxed{4049}$.

Remark: for a rigorous proof, see below.

Solution 2: We can calculate the expected number of coins David will collect on the n^{th} move. After n moves, David will be at some point $(1+k, 1+n-k)$, where $0 \le k \le n$ if $n \le 2023$ and $0 \le k \le 2023$ if $n > 2023$. Notice that he collects a coin with probability $\frac{1+k}{2+n}$, and more importantly that the denominator remains constant across all possible destinations after n moves. Thus, the expected value of coins David will collect on move n is

$$\frac{1}{d} \sum_{k=k_1}^{k_d} \frac{1+k}{2+n}$$

where d is the number of possible destinations after n moves, and $k_1, k_2, \dots, k_d$ is the number of horizontal moves required to reach each of these destinations. Notice that for $n \le 2023$, we have $d = n+1$ and $k_i = i-1$ for all $1 \le i \le d$, and for $n > 2023$, we have $d = 4047 - n$ and $k_i = 2024 - i$ for all $1 \le i \le d$. Thus, for

$n \le 2023$ we can rewrite the expected value of coins collected on move n as

$$\begin{aligned}\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1+(i-1)}{2+n} &= \frac{1}{n+1}\left(\frac{1}{n+2}+\frac{2}{n+2}+\cdots+\frac{n+1}{n+2}\right)\\ &= \frac{1}{n+1}\frac{1}{n+2}\left(1+2+\cdots+(n+1)\right)\\ &= \frac{1}{n+1}\frac{1}{n+2}\frac{(n+1)(n+2)}{2}\\ &= \frac{1}{2}\end{aligned}$$

For $n > 2023$, we can rewrite the expected value of coins collected on move n as

$$\begin{aligned}\frac{1}{4047-n}\sum_{i=1}^{4047-n}\frac{1+(2024-i)}{2+n} &= \frac{1}{4047-n}\left(\frac{2024}{n+2}+\frac{2023}{n+2}+\cdots+\frac{n-2022}{n+2}\right)\\ &= \frac{1}{4047-n}\frac{1}{n+2}\left(2024+2023+\cdots+(n-2022)\right)\\ &= \frac{1}{4047-n}\frac{1}{n+2}\left(\frac{2024\cdot 2025}{2}-\frac{(n-2023)(n-2022)}{2}\right)\\ &= \frac{1}{4047-n}\frac{1}{n+2}\left(\frac{-n^2+4045n+8094}{2}\right)\\ &= \frac{1}{4047-n}\frac{1}{n+2}\left(\frac{(4047-n)(n+2)}{2}\right)\\ &= \frac{1}{2}\end{aligned}$$

Thus, regardless of n, the expected value of coins collected on move n is $\frac{1}{2}$. Since there are 4047 different values of n (0 through 4046), the expected number of coins David will collect on his journey is $4047 \cdot \frac{1}{2}$, so the answer is $4047 + 2 = \boxed{4049}$.

T8. Points A, B, and C lie on line ℓ in that order such that $AB = 24$ and $BC = 31$. Let D and E be points lying on the same side of ℓ such that ΔABD and ΔBCE are equilateral triangles. Let F be the point closer to ℓ such that ΔDEF is an equilateral triangle. Find the value of $AF + BF + CF$.

Proposed by: Anthony Yang

Answer: $\boxed{62}$

Solution: Notice that $\angle DBE = 180^\circ - \angle ABD - \angle CBE = 60^\circ = \angle DFE$. This means that $\angle DBE$ and $\angle DFE$ subtend the same arc on the same chord DE, so D, B, F, E must lie on the same circle. Thus, $DBFE$ is a cyclic quadrilateral. From the properties of a cyclic quadrilateral, we know that $\angle DEF + \angle DBF = 180^\circ$, but since ΔDEF is equilateral we also know that $\angle DEF = 60^\circ$, so $\angle DBF = 120^\circ$. Notice that

$$120^\circ = \angle DBF = \angle DBE + \angle FBE = 60^\circ + \angle FBE \implies \angle FBE = 60^\circ$$

However, we know that $\angle CBE = 60^\circ = \angle FBE$, and from inspection we know that F cannot lie on line BD. Thus, we find that F must lie on BC, meaning A, B, F, C are collinear in that order. It now suffices to find the length of BF. Let $DE = EF = FD = x$ and $BF = y$. Using Ptolemy's Theorem on $DBFE$, we get

$$BD \cdot EF + BF \cdot DE = DF \cdot BE \implies 24x + xy = 31x$$

Dividing both sides by x gives $24 + y = 31$ which means $y = 7$. Thus, we have

$$AF + BF + CF = (AB + BF) + BF + CF = 31 + 7 + 24 = \boxed{62}$$

T9. Let $f(k)$ denote the k^{th} smallest positive integer that is not a perfect square. For example, $f(1) = 2$, $f(2) = 3$, and $f(3) = 5$. Let n be the positive integer that satisfies

$$f^{2024}(n) = 2024^2 + 1$$

where f^{2024} denotes the function f composed 2024 times. Find the remainder when n is divided by 1000.

Proposed by: Harry Kim

Answer: $\boxed{145}$

Solution: Let $g(k)$ denote the number of perfect squares less than k. For example, $g(5) = 2$ and $g(99) = 9$. Then, if $f(k) = n$, we have

$$k = n - g(n)$$

We can now work backwards from $f^{2024}(n)$ to find n. We have

$$f^{2024}(n) = f(f^{2023}(n)) = 2024^2 + 1 \implies f^{2023}(n) = 2024^2 + 1 - g(2024^2 + 1)$$

Since $g(2024^2 + 1) = 2024$, we have $f^{2023}(n) = 2024^2 - 2023$. Similarly, we find

$$f^{2022}(n) = 2024^2 - 2023 - g(2024^2 - 2023)$$

Notice that $2023^2 < 2024^2 - 2023 < 2024^2$ so $g(2024^2 - 2023) = 2023$. We now have $f^{2022}(n) = 2024^2 - 4046$. However, observe that

$$2024^2 - 2023^2 = (2024 - 2023)(2024 + 2023) = 4047$$

so we can rewrite $f^{2022}(n)$ as

$$f^{2022}(n) = 2024^2 - 4046 = (2024^2 - 4047) + 1 = 2023^2 + 1$$

From similar reasoning, we find that $f^{2020}(n) = 2022^2 + 1$ and in general that

$$f^{2024-2a}(n) = (2024 - a)^2 + 1$$

for all $0 \le a \le 1012$. Since we wish to find n, we can let $a = 1012$ to get

$$f^0(n) = n = (2024 - 1012)^2 + 1 = 1012^2 + 1$$

Taking this mod 1000, we get

$$n \equiv 1012^2 + 1 \equiv 12^2 + 1 \equiv \boxed{145} \pmod{1000}$$

T10. Anika and Angela are playing a game of tic-tac-toe. They take turns drawing shapes in a 3×3 grid. Anika goes first and draws an X each turn and Angela draws an O each turn. The first player who gets three of their shapes in a line (vertical, horizontal, diagonal) wins the game. Suppose that Anika won the game. Compute the remainder of the number of possible final configurations of the 3×3 grid divided by 1000.

X	O	
X		O
X	O	X

Proposed by: Harry Kim

Answer: $\boxed{630}$

Solution:

We can divide the problem into three cases.

a) **Case 1:** The game ends with three Xs on the board.

We simply choose a line for the Xs and choose the placement of O randomly. This gives $8 \cdot \binom{6}{2} = 120$ configurations.

b) **Case 2:** There are four Xs on the board.

If the Xs are on a horizontal or vertical line, we need to make sure the three Os are not in a line. This gives $6\left(\binom{6}{3} - 2\right) \cdot 3 = 324$ configurations. In the case of the two diagonals, the Os cannot make a line, so we get $2 \cdot 6 \cdot \binom{5}{3} = 120$ configurations.

c) **Case 3:** Five Xs on the board.

In the final case, first consider when Xs make a horizontal or vertical line without any additional lines. This gives $6(9 - 5) = 24$ configurations. If the Xs make a diagonals without additional lines, there are $2\left(\binom{6}{2} - 5\right) = 20$ configurations. Finally when the Xs make two different kinds of lines, there are $\binom{8}{2} - 3 - 3 = 22$ configurations, subtracting the (horizontal, horizontal) and (vertical, vertical) pairs.

Thus, the answer is $120 + 324 + 120 + 24 + 20 + 22 = \boxed{630}$.

T11. Let $\overline{abcdef}$ be the unique six-digit number with non-zero digits such that

$$(\overline{abc} - \overline{def})^2 = \overline{abcdef}$$

Find the value of $\overline{abcdef}$.

Proposed by: Anthony Yang and Brandon Xu

Answer: $\boxed{132496}$

Solution: Let the three digit number $\overline{abc}$ be x, and $\overline{def}$ be y. We have

$$(x - y)^2 = 1000x + y$$

which can be rearranged to

$$y^2 - (2x + 1)y - 1000x + x^2 = 0$$

By quadratic formula on y, we get

$$y = \frac{(2x + 1) + \sqrt{4004x + 1}}{2}$$

Note that since x is a three-digit number, we must take the positive sign in the quadratic formula for y to be positive. Further, by approximating $y \approx x + \sqrt{1001x}$, we find that $x < 400$ in order for $y < 1000$. For y to be integer, we must have $4004x + 1 = z^2$ for some positive integer z. This can be rewritten as

$$z^2 - 1 = 4004x \implies (z - 1)(z + 1) = 4004x$$

Since the RHS is divisible by 4, we know that both $z-1$ and $z+1$ are even. Now let $k = \frac{z-1}{2}$. We have $k(k+1) = 1001x$. Notice that

$$k^2 < k(k+1) = 1001x < 1001 \cdot 400 < 1024 \cdot 400 = (640)^2$$

so we now have that $k < 640$. Since $1001 = 7 \cdot 11 \cdot 13$, the primes 7, 11, and 13 each have to divide either k or $k+1$. Note that all three primes cannot divide one of k or $k+1$, or else x would be a four-digit number. Thus, k contains either one or two of 7, 11, and 13. We can do casework on the combination of prime divisors of k, $k+1$.

a) **Case 1:** 7 divides k, 143 divides $k+1$.

   Since $143 \equiv 3 \pmod 7$, the only value of k is $5 \cdot 143 - 1 = 714$. This is greater than 640, so there are no solutions for this case.

b) **Case 2:** 143 divides k, 7 divides $k+1$.

   We get $k = 286$ which gives $x = 82$, which is not a three-digit integer. Thus, there are no solutions for this case.

c) **Case 3:** 13 divides k, 77 divides $k+1$.

   Since $77 \equiv -1 \pmod{13}$, we find that $k = 77 \cdot 12 - 1 = 923$. This is greater than 640, so there are no solutions for this case.

d) **Case 4:** 77 divides k, 13 divides $k+1$.

   Clearly, we have $k = 77$. This yields $x = 6$ which is not a three-digit number, so there are no solutions for this case.

e) **Case 5:** 91 divides k, 11 divides $k+1$.

   Since $91 \equiv 3 \pmod{11}$, we find that $k = 91 \cdot 7 = 637$. While this is less than 640, this yields $x = 406$, which is greater than 400. Thus, there are no solutions for this case.

f) **Case 6:** 11 divides k, 91 divides $k+1$.

   We can easily find $k = 363$, which gives $x = 132$. Recall that $\sqrt{4004x+1} = z = 2k+1 = 727$. Solving for y gives

   $$y = \frac{(2 \cdot 132 + 1) + 727}{2} = 496$$

   which is the unique solution.

Since $x = 132$ and $y = 496$, our answer is $\boxed{132496}$.

T12. For a positive integer n, let $f_k(n)$ denote the number of positive divisors of n less than or equal to k. Suppose two (not necessarily distinct) positive integers a, b are randomly chosen from the set $\{1, 2, \cdots 1200\}$. If the expected value of $f_6(ab)$ can be expressed as $\frac{a}{b}$ where a and b are relatively prime positive integers, find $a+b$.

Proposed by: Harry Kim

Answer: $\boxed{1031}$

Solution: Let p_i be the probability that i is a divisor of ab. By linearity of expectation, the expected value of $f_6(ab)$ is $p_1 + p_2 + p_3 + p_4 + p_5 + p_6$. It is easy to see that $p_1 = 1$ as 1 is always a divisor of a positive integer. Observe that 2 is not a divisor of ab if a and b are both odd so $p_2 = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$. Similarly, $p_3 = 1 - \left(\frac{2}{3}\right)^2 = \frac{5}{9}$ and $p_5 = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25}$. For p_4, notice that 4 is not a divisor of

ab if a and b are both odd or one of a or b is odd and the other is a multiple of 2 that is not a multiple of 4. Therefore, $p_4 = 1 - \left(\frac{1}{2}\right)^2 - 2 \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{2}$. For p_6, we use principle of inclusion exclusion to see that $p_6 = 1 - \left(\frac{1}{2}\right)^2 - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{6}\right)^2 = \frac{5}{12}$ since for modulo 6, there are exactly 2 values that are relatively prime to 2 and 3. Therefore, the expected value is $1 + \frac{3}{4} + \frac{5}{9} + \frac{9}{25} + \frac{1}{2} + \frac{5}{12} = \frac{806}{225}$ so our answer is $806 + 225 = \boxed{1031}$.

T13. A ladybug is in the center of a cube with side length 4, such that it is distance 2 away from every face of the cube. The ladybug wants to touch four faces of the cube and return to its original position (the ladybug "touches" a face of the cube if it is at any interior or boundary point of the face). If M is the minimum distance the ladybug needs to travel, find M^2.

Proposed by: Harry Kim

Answer: $\boxed{96}$

Solution: Reflect the starting point C (center) across some four faces of the cube to point C'. Then CC' would be the minimum distance to touch the selected four faces and come back to the center. There are two ways to choose four faces on a cube that when rotated are distinct. Choosing four faces in a loop will yield minimum distance $8\sqrt{2}$. Choosing four faces centered on one edge will yield minimum distance $\sqrt{8^2 + 4^2 + 4^2} = \sqrt{96}$. Since $\sqrt{96} < 8\sqrt{2}$, the answer is $\boxed{96}$.

T14. Cindy and Zadie are playing a game. There are three piles of candies, A, B, C, each with a, b, c candies respectively where $1 \le a, b, c \le 8$. Cindy begins the game and each turn, a player chooses a pile and eats any positive number of candies in that pile. The player who eats the last remaining candy loses. What is the number of triples (a, b, c) such that Cindy can guarantee a win no matter how Zadie plays?

Proposed by: Harry Kim

Answer: $\boxed{469}$

Solution: Say that a player is in the *winning state* if they can play a move to make the other player in the losing state and in the *losing state* if no matter what they play, they will make the other player in the winning state. First, observe that $(0, n, n)$ is a losing state when $n \ge 2$. From there, we can construct losing states by having no pairs of numbers overlap with previous losing states. We find that the only losing states are $(1, 1, 1), (1, 2, 3), (1, 4, 5), (1, 6, 7), (2, 4, 6), (2, 5, 7), (3, 4, 7), (3, 5, 6)$. Notice that $(1, 1, 1)$ has only 1 permutation and the rest have 6 permutations. Hence the triples that Anthony can guarantee a win is $8^3 - 1 - 42 = \boxed{469}$.

T15. Two circles ω_1 and ω_2 with radii 16 and 3, respectively, are internally tangent at point A. Let B and C be points on ω_1 such that BC is tangent to ω_2 and $\angle BAC = 120^\circ$. The value of $AB + AC$ can be expressed as $\frac{a\sqrt{b}}{c}$ where a and c are relatively prime positive integers and b is square-free. Find $a + b + c$.

Proposed by: Harry Kim

Answer: $\boxed{116}$

Solution: Let O_1, O_2 denote the centers of ω_1 and ω_2, respectively, and let D be the point where ω_2 meets AC. Let M denote the midpoint of the arc AC not containing A. By homothety, since $O_1M \parallel O_2D$, we see that A, D, M are collinear (this is otherwise known as shooting lemma). Since $\angle BMC = 60^\circ$ and $BM = CM$, it follows that $\triangle BMC$ is an equilateral triangle with side length $16\sqrt{3}$. Notice that

O_1M and O_2D are parallel, so $\triangle AO_2D \sim \triangle AO_1M$. Therefore, $\frac{AD}{AM} = \frac{AO_2}{AO_1} = \frac{3}{16}$. Let $AD = 3x$ and $AM = 16x$. Then $DM = 16x - 3x = 13x$. Observe that $\angle MCD = \angle MAC = 60^\circ$ so $MC^2 = MD \cdot MA$. Therefore, $13x \cdot 16x = 48$, so $x = \frac{4\sqrt{39}}{13}$, and $AM = 16x = \frac{64\sqrt{39}}{13}$. By Ptolemy's theorem, we find $AB+AC = AM$, so $AB + AC = \frac{64\sqrt{39}}{13}$. Thus the answer is $64 + 39 + 13 = \boxed{116}$.