MOAA 2025 Accuracy Round Solutions

Math Open at Andover

October 11, 2025

A1. Find the remainder when

$$1! + 2! + 3! + \cdots + 2025!$$

is divided by 45.

Proposed by: Paige Zhu

Answer: $\boxed{18}$

Solution: Since $6! = 720 = 45 \cdot 16$, we know that the sum of $6! + 7! + \cdots + 2025!$ has remainder 0 when divided by 45. So, we can simply calculate

$$1! + 2! + 3! + 4! + 5! = 153 = 45 \cdot 3 + \boxed{18}.$$

A2. Niki is baking cookies for X-Camp's bake sale at a rate of n cookies per minute, where n is a positive integer. After 20 minutes, the total number of cookies she has is a multiple of 12. She bakes for 5 more minutes, then accidentally drops 26 cookies on the way to the sale. If she arrives at the sale with a prime number of cookies, what is the smallest possible number of cookies she could have?

Proposed by: Paige Zhu

Answer: $\boxed{199}$

Solution: Since $20n$ is a multiple of 12, n must be a multiple of 3. Writing $n = 3k$, the total number of cookies she ends up bringing to the sale is just $25 \cdot 3k - 26 = 75k - 26$. Trying small values of k gives:

$$k = 1 \implies 75k - 26 = 49,$$
$$k = 2 \implies 75k - 26 = 124,$$
$$k = 3 \implies 75k - 26 = 199.$$

Clearly, 49 and 124 are not prime, and a bit of arithmetic shows that $\boxed{199}$ is prime, as desired.

A3. A positive integer with n digits is formed using only the digits 1, 2, and 3. The integer is called *even-friendly* if the sum of every pair of adjacent digits is even. Find the number of even-friendly 10-digit integers.

Proposed by: Paige Zhu

Answer: $\boxed{1025}$

Solution: The parity (even or odd) of each digit is uniquely determined from the parity of the first digit. If the first digit is odd, then every digit must be odd, which can happen in $2^{10} = 1024$ ways, since there are two choices for each digit. Otherwise, if the first digit is even, then every digit is even, which can only happen in 1 way (if every digit is 2). Hence, the answer is $1024 + 1 = \boxed{1025}$.

A4. How many ways are there to permute the 8 letters in "MATHDASH" such that the letters in the odd positions appear in alphabetical order from left to right, and the letters in the even positions also appear in alphabetical order from left to right?
Proposed by: Paige Zhu

Answer: $\boxed{36}$

Solution: Reorder MATHDASH in alphabetical order to get AADHHMST. We wish to partition the set of letters into two sets of 4 letters, corresponding to the letters in the even and odd spots. Once the letters for the even (or odd) spots are chosen, the resulting permuted word is determined uniquely. So, the answer is simply the number of distinct groups of 4 letters selected from the collection $\{A, A, H, H, D, M, S, T\}$. We consider the following three cases:

Case 1: All four letters are distinct.

We select 4 letters from the set $\{A, H, D, M, S, T\}$. This yields $\binom{6}{4} = 15$ groups.

Case 2: One letter occurs twice, and two letters occur once.

There are two ways to choose the letter to be duplicated (either A or H), and the remaining 2 letters can be chosen in $\binom{5}{2} = 10$ ways, for 20 total groups.

Case 3: Two letters which both occur twice.

Both A and H must be used twice, giving 1 possible group.

The answer is $15 + 20 + 1 = \boxed{36}$.

A5. Two circles ω_1 and ω_2 with radius 4 are centered at A and B, respectively, so that B lies on ω_1 and A lies on ω_2. Let points C and D be the intersections of ω_1 and ω_2. Point P is on ω_1 such that $\angle APC = 45^\circ$, and Q is the intersection of PC and ω_2 not at C. If lines PA and QB intersect at E, the value of DE^2 can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Brandon Xu

Answer: $\boxed{19}$

Solution:

Since $\triangle ABD$ and $\triangle ABC$ are equilateral, we have $\angle CAD = \angle CBD = 120^\circ$. Since $PDBC$ is a cyclic quadrilateral, we have $\angle QPD = \angle CPD = 180^\circ - \angle CBD = 60^\circ$. Then, $\angle APD = 15^\circ$. We can angle chase to find that

$$\angle BCQ = 180^\circ - \angle PCA - \angle ACB = 75^\circ$$

and

$$\angle ABQ = \angle ABC + \angle CBQ = 60^\circ + 180^\circ - 2\angle BCQ = 90^\circ.$$

Hence, ABE is a right triangle, with $\angle BAE = 30^\circ$. Thus, we have that

$$BE = \frac{AB}{\sqrt{3}} = \frac{4}{\sqrt{3}}.$$

Since PA is a perpendicular bisector of BD and E lies on PA, we must have $DE = BE$, so the answer is

$$DE^2 = \left(\frac{4}{\sqrt{3}}\right)^2 = \frac{16}{3} \implies m + n = \boxed{19}.$$

A6. The 2025 MOAA board consists of 5 directors and 5 associates. Two directors and one associate have names beginning with E. In how many ways can all 10 members be seated around a circular table so that no two directors are adjacent, no two associates are adjacent, and no two people whose names begin with E are adjacent?

Proposed by: Brandon Xu

Answer: $\boxed{864}$

Solution: Because no two directors are adjacent and no two associates are adjacent, the directors and associates must alternate around the circle. We can fix the directors' spots, accounting for rotation, in $(5-1)! = 24$ ways. Let D_1 and D_2 denote the two directors whose names start with E, and A denote the associate whose name starts with E. We have two possible cases based on the placement of D_1 and D_2.

Case 1: D_1 and D_2 are adjacent in the directors' circular order.

The directors can be arranged in $2 \cdot 3! = 12$ ways. Then, A cannot sit in 3 of the 5 gaps between directors, and the remaining 4 associates can be permuted freely. So, there are $12 \cdot 2 \cdot 24 = 576$ arrangements in this case.

Case 2: D_1 and D_2 are not adjacent in the directors' circular order.

The directors can be arranged in $24 - 12 = 12$ ways. Note that A can only sit in one gap between the directors. The remaining associates can still be arranged in $4! = 24$ ways, so we have $12 \cdot 24 = 288$ arrangements in this case.

So, the total number of valid seatings is $576 + 288 = \boxed{864}$.

A7. Find the smallest positive integer value of k for which the sum of the values of n such that $n^2 + k$ is divisible by $kn + 1$ is greater than 20252025.

Proposed by: Brandon Xu and Jialai She

Answer: $\boxed{4500}$

Solution: Let $d = kn + 1$. Then, since $n = \frac{d-1}{k}$, we have

$$n^2 + k \equiv \left(\frac{-1}{k}\right)^2 + k \equiv \frac{1+k^3}{k^2} \pmod{d}.$$

Then, we know that

$$\frac{1+k^3}{k^2} \equiv 0 \pmod{d}$$

which implies that $kn+1$ is a factor of k^3+1. Hence, the valid solutions n correspond to divisors d of k^3+1 with $d \equiv 1 \pmod{k}$. We can factor $k^3+1 = (k+1)(k^2-k+1)$.

We do casework on the size of n.

Case 1: $n \geq k$. Then, $d = kn + 1 \geq k^2 + 1$. Note that

$$d \mid nd - k(n^2 + k) \implies d \mid n - k^2.$$

However, since $d \leq k^3 + 1$ implies $n < k^2 + 1$, we must have $|n - k^2| < d$. Thus, $n - k^2 = 0$. and $n = k^2$ is the only solution.

Case 2: $1 \leq n \leq k - 1$. Then, we have that $k + 1 \leq d \leq k(k-1) + 1 = k^2 - k + 1$. Clearly, both $n = 1$ and $n = k-1$ work, coresponding to $d = k+1$ and $d = k^2-k+1$. Now, we will show that there are no valid solutions when $1 < n < k - 1$. Since

$n > 1$, we must have $d \mid k^2 - k + 1$. From $d = kn + 1$, we have that $kn = -1 \pmod{d}$ so $k = -n^{-1} \pmod{d}$. Then, we have

$$k^2 - k + 1 \equiv n^{-2} + n^{-1} + 1 \equiv \frac{n^2 + n + 1}{n^2} \equiv 0 \pmod{d}$$

$$\implies n^2 + n + 1 \equiv 0 \pmod{d}.$$

However, since $n < k - 1$, we have

$$0 < n^2 + n + 1 < d$$

which clearly is a contradiction.

Hence, the only valid solutions for n are 1, $k-1$, and k^2. So, we need $1+(k-1)+k^2 = k(k+1) > 20252025$. Since $4500^2 \leq 20252025 \leq 4500 \cdot 4501$, the smallest possible value of k is $\boxed{4500}$.

A8. Let ABC be a triangle with $AB = 7$, $AC = 3\sqrt{14}$, and $BC = 14$. Let D be a point on BC such that $CD = 4$, and E be a point on ray $\overrightarrow{AD}$ such that $DE = 5$. Define points O, P, and Q as the circumcenters of triangles ABC, ABD, and CDE, respectively. Given that $\sqrt{OD^2 + PQ^2}$ can be written as $\frac{m\sqrt{n}}{p}$ where m and p are relatively prime integers, and n is not divisible by the square of any prime, find $m + n + p$.

Proposed by: Brandon Xu

Answer: $\boxed{111}$

Solution: Using Stewart's Theorem on cevian AD in ABC gives

$$14(AD^2 + 4 \cdot 10) = (3\sqrt{14})^2 \cdot 10 + 7^2 \cdot 4.$$

Solving gives $AD = 8$. Then, note that $CD \cdot DB = 4 \cdot 10 = 8 \cdot 5 = AD \cdot DE$. Hence, $ACEB$ is cyclic by Power of a Point. Angle chasing gives $\angle QDE = 90^\circ - \angle DCE = 90^\circ - \angle DAB$. Hence, $QD \perp AB$, which gives $QD \parallel OP$. Similarly, we can see that $DP \parallel QO$. Hence, $OPDQ$ is a parallelogram. Thus, we have that

$$OD^2 + PQ^2 = 2(QD^2 + DP^2).$$

However, since $\triangle ABD \sim \triangle CDE$, and $\frac{AD}{CD} = \frac{1}{2}$, we have $DP = 2QD$ and $CE = \frac{7}{2}$. By Heron's formula on $\triangle CDE$, we have

$$[CDE] = \sqrt{\frac{25}{4} \cdot \frac{11}{4} \cdot \frac{9}{4} \cdot \frac{5}{4}} = \frac{15\sqrt{55}}{16}.$$

Thus, we can find

$$QD = \frac{CD \cdot DE \cdot EC}{4[CDE]} = \frac{56\sqrt{55}}{165}.$$

So, we have

$$\begin{aligned}\sqrt{OD^2 + PQ^2} &= \sqrt{2(QD^2 + DP^2)} \\ &= \sqrt{2(QD^2 + 4 \cdot QD^2)} \\ &= \sqrt{10} \cdot QD \\ &= \frac{56\sqrt{22}}{33}\end{aligned}$$

The answer is $56 + 22 + 33 = \boxed{111}$.

A9. Mr. DoBa is using a strange calculator from Citadel that can only display the last two digits of any number. For example, the number 2025 would appear as 25. There are three buttons: button A multiplies the displayed number by 2, button B multiplies the displayed number by 3, and button C multiplies the displayed number by 7. If the calculator currently displays 32, how many ways can Mr. DoBa press 9 buttons so that the screen displays 24 at the end?

Proposed by: Brandon Xu

Answer: $\boxed{1975}$

Solution: The calculator shows the remainder modulo 100 after every action. Hence, we want the final result to be equivalent to $24 \pmod{100}$. We will consider mod 4 and mod 25 separately. Note that no matter the sequence of button presses, the number on the screen will always be a multiple of 4, since 4 divides 32, and taking the remainder mod 100 does not affect whether the result is divisible by 4. Let

$$M = 2^a 3^b 7^c$$

be the number multiplied to 32 after 9 button presses, where button A is pressed a times, button B is pressed b times, and button C is pressed c times. For the sequence of button presses to be valid, we must have

$$32M \equiv 24 \pmod{25}$$
$$7M \equiv 24 \pmod{25}$$

By inspection, we can see that $M \equiv 7 \pmod{25}$ is the only solution. By Euler's theorem, $2^{20} \equiv 1 \pmod{25}$, since $\phi(25) = 20$, where $\phi(n)$ is the Euler totient function. Moreover, note that $2^{10} \not\equiv 1 \pmod{25}$ and $2^4 \not\equiv 1 \pmod{25}$. Working in mod 25, notice that pressing button B once is equivalent to pressing button A 7 times, since $2^7 \equiv 3 \pmod{25}$. Similarly, pressing button C once is equivalent to pressing button A 5 times, since $2^5 \equiv 7 \pmod{25}$. Hence, we can also express M as:

$$M \equiv 2^{a+7b+5c} \equiv 2^5 \pmod{25}.$$

As we just showed that $2^{20} \equiv 1 \pmod{25}$, we must have $a + 7b + 5c \equiv 5 \pmod{20}$, and $a + b + c = 9$. Since a, b, and c are non-negative, we must have that

$$9 \leq a + 7b + 5c \leq 63,$$

so either $a + 7b + 5c = 25$, or $a + 7b + 5c = 45$. Since $a + b + c = 9$, these equations become $6b + 4c = 16$ and $6b + 4c = 36$.

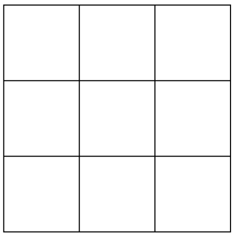
Now, we can list the solutions:

$$(a, b, c) \in \{(5, 0, 4), (6, 2, 1), (3, 6, 0), (2, 4, 3), (6, 2, 1), (0, 0, 9)\}.$$

Finally, to get the number of sequences of button presses, we just sum the number of ways to arrange a A's, b B's, and c C's in a row, over all possible choices of (a, b, c). We get

$$\binom{9}{4} + 9\binom{8}{2} + \binom{9}{3} + \binom{9}{2}\binom{7}{3} + 9\binom{8}{2} + 1 = 126 + 9 \cdot 28 + 84 + 36 \cdot 35 + 9 \cdot 28 + 1$$
$$= \boxed{1975}.$$

A10. Let N be the number of ways to fill a 3×3 grid with non-negative integers such that the sum of the numbers in each row and each column is equal to 18. Find the greatest prime divisor of N.



Proposed by: Brandon Xu

Answer: $\boxed{191}$

Solution 1: Every valid grid can be represented as shown below, where a, b, c, d, e, f are non-negative integers, and $a + b + c + d + e + f = 18$:

$a + d$	$b + e$	$c + f$
$b + f$	$c + d$	$a + e$
$c + e$	$a + f$	$b + d$

This inspires us to represent such valid grids using 6-tuples (a, b, c, d, e, f). However, note that if a, b, c, d, e and f are all positive, the 6-tuple $(a-1, b-1, c-1, d+1, e+1, f+1)$ represents the same grid. Thus, to avoid overcounting, we must impose the additional condition that at least one of a, b, c must be zero.

We can count the number of valid 6-tuples using Stars and Bars. If there is no restriction on the elements, other than that they must be non-negative, there are $\binom{23}{5}$ possible 6-tuples. However, we must subtract the case where a, b, c are all greater than 0, which occurs in $\binom{20}{5}$ ways. So, we have

$$N = \binom{23}{5} - \binom{20}{5}.$$

Expanding and simplifying gives:

$$\begin{aligned}\binom{23}{5} - \binom{20}{5} &= \frac{23 \cdot 22 \cdot 21 \cdot 20 \cdot 19}{120} - \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{120} \\ &= 23 \cdot 11 \cdot 7 \cdot 19 - 19 \cdot 3 \cdot 17 \cdot 16 \\ &= 19 \cdot (23 \cdot 11 \cdot 7 - 3 \cdot 17 \cdot 16) \\ &= 19 \cdot 955 \\ &= 5 \cdot 19 \cdot 191\end{aligned}$$

The largest prime divisor of N is $\boxed{191}$.

Solution 2: Note that the four entries represented by a, b, c and d are enough to determine the rest of the grid:

a	b	$18 - a - b$
c	d	$18 - c - d$
$18 - a - c$	$18 - b - d$	$\cdots$

In particular, we can calculate the bottom right entry to be equal to $a+b+c+d-18$.

For each entry to be non-negative, the following equations must all hold:

$$a + b \leq 18, \quad c + d \leq 18, \quad a + c \leq 18, \quad b + d \leq 18, \quad a + b + c + d \geq 18.$$

First, we will ignore the constraints that $a + c \leq 18$ and $b + d \leq 18$, and just count the number of quadruples (a, b, c, d) satisfying $a + b \leq 18$ and $c + d \leq 18$. By Stars and Bars, there are $\binom{18+2}{2} = 190$ choices for a, b, and similarly for c, d, giving 190^2 quadruples.

Next, we remove the quadruples in which $a + c \geq 19$ or $b + d \geq 19$. Notice however, that it is impossible for both $a + c \geq 19$ and $b + d \geq 19$ to hold at the same time, since $a + b + c + d = (a + b) + (c + d) \leq 36$. WLOG, we can assume that $a + c \geq 19$, and double the result for $b + d \geq 19$. Notice that for a fixed a, the value of c can range from $19 - a$ to 18, inclusive. Once (a, c) are found, there are just $(19 - a)(19 - c)$ such quadruples by freely choosing (b, d) such that $a + b \leq 18$ and $c + d \leq 18$. So, for a fixed value of a, the value of $(19 - c)$ ranges from 1 to a. Hence, we must subtract

$$(19 - a)\frac{a(a + 1)}{2}$$

quadruples for each value of a. We have:

$$\begin{aligned} 2 \cdot \sum_{a=1}^{18} (19 - a)\frac{a(a + 1)}{2} &= \sum_{a=1}^{18} -a^3 + 18a^2 + 19a \\ &= -(9 \cdot 19)^2 + 18 \cdot \frac{18 \cdot 19 \cdot 37}{6} + 19 \cdot \frac{18 \cdot 19}{2} \\ &= 11970 \end{aligned}$$

Finally, we must also subtract the case when $a + b + c + d < 18$. This is equivalent to the number of solutions to $a + b + c + d + e = 18$ where $e > 0$. By Stars and Bars, there are $\binom{21}{4} = 5985$ quadruples in this case. So, the number of valid grids is $36100 - 11970 - 5985 = 18145$. Factoring gives

$$18145 = 5 \cdot 19 \cdot \boxed{191}.$$