MOAA 2025 Gunga Bowl Solutions

Math Open at Andover

October 11th, 2025

G1. Evaluate

$$\frac{\sqrt{2025}}{5} + \frac{2025^2}{\sqrt{2025^3}}.$$

Proposed by: Paige Zhu

Answer: $\boxed{54}$

Solution: Note that $\sqrt{2025} = 45$. Straightforward calculation gives $\frac{45}{5} + \frac{2025^2}{45^3} = \boxed{54}$.

G2. Paige has 5 times as many labubus as Bill. If Bill steals 3 of Paige's labubus, Paige will only have 3 times as many labubus as Bill. How many labubus did Paige have originally?

Proposed by: Paige Zhu

Answer: $\boxed{30}$

Solution: Set the number of labubus Paige had originally to be $5x$. We can rewrite the question into the following system of equation:

$$3(x + 3) = 5x - 3$$

Solving, we get $x = 6$, so $5x = \boxed{30}$.

G3. Angela wants to choose 3 Alphastar classes from a catalog of 3 math classes, 4 physics classes. and 2 computer science classes. She wants to choose classes from at least two subject areas, and she must choose a math class. How many ways she can select her classes?

Proposed by: Brandon Xu

Answer: $\boxed{63}$

Solution: Angela must take either one or two math classes. We will do casework on the number of math classes she takes:

Case 1: Angela takes one math class. She can choose her math class in 3 ways. Then, she can select her two other classes freely from the six remaining classes in $(6 \cdot 5)/2 = 15$ ways. In total, this gives $3 \cdot 15 = 45$ ways.

Case 2: Angela takes two math classes. She can choose the two math classes in 3 ways. Then, she has 6 choices for her last class, giving 18 combinations.

Hence, she can select her classes in $18 + 45 = \boxed{63}$ ways.

G4. While waiting for his Expii class, Gunga begins drawing a square of side length 1, then a circle of radius 1, then a square of side length 2, then a circle of radius 2, and so on. In general, for each positive integer k, he draws a square of side length k followed by a circle of radius k. He continues this process until the total area of all the shapes he has drawn exceeds 5000. How many shapes does Gunga draw in total?

Proposed by: Brandon Xu

Answer: $\boxed{30}$

Solution: For each integer k, the sum of the areas of the square with side length k and the circle with radius k is

$$k^2 + \pi k^2 = (\pi + 1)k^2.$$

First, we want the smallest integer k such that

$$\sum_{k=1}^{n} (\pi + 1)k^2 > 5000.$$

Using $\pi \approx 3.1$, we need

$$\sum_{k=0}^{n} k^2 > \frac{5000}{4.1} \approx 1219.$$

Using the formula for the sum of squares, we have

$$\frac{n(n+1)(2n+1)}{6} > 1219.$$

Some trial and error reveals that $n = 15$ gives

$$\sum_{k=0}^{15} k^2 = 1240 > 1219.$$

Then, it is fairly easy to see that both the square and the circle is necessary for the area to exceed 5000, so the answer is $\boxed{30}$.

G5. Suppose $\underline{a}\,\underline{b}_7$ is a two-digit base-7 number with $a \neq 0$, such that its value in base 10 is equal to the two-digit number $\underline{b}\,\underline{a}_{10}$ obtained by reversing its digits. Find the sum of all such numbers when expressed in base 10.

Proposed by: Paige Zhu

Answer: $\boxed{69}$

Solution: The base 10 value of $\underline{a}\,\underline{b}_7$ is $b + 7a$. Since this is equal to $\underline{b}\,\underline{a}_{10}$, we must have

$$b + 7a = a + 10b \implies 2a = 3b.$$

Then, the possible solutions for (a, b) are $(3, 2)$ and $(6, 4)$, since we must have $a, b < 7$. Thus, the answer is $23 + 46 = \boxed{69}$.

G6. At the MehtA+ summer camp, there are 2025 students, and numerous clubs. Each student can join any number of clubs, but no two students can be in exactly the same set of clubs. What is the minimum number of clubs at the summer camp?

Proposed by: Paige Zhu

Answer: $\boxed{11}$

Solution: The key realization is that when there are more students than subsets of clubs, there must exist two students that are in exactly the same clubs. If there are only 10 clubs at the camp, then there is a total of $2^{10} = 1024$ unique subsets of clubs, which is less than 2025. When there are 11 clubs, there are $2^{11} = 2048 > 2025$ unique subsets, and hence, $\boxed{11}$ clubs is the minimum.

G7. A quadrilateral is drawn so that all four vertices lie on lattice points, and no other lattice points lie on any of its sides. If the quadrilateral has an area of 18 square units, how many lattice points lie strictly within its boundaries?

Proposed by: Paige Zhu

Answer: $\boxed{17}$

Solution: We can apply Pick's theorem, which states that the area, A, of a polygon with vertices at lattice points can be related to the number of lattice points on the boundary, B, and the number of lattice points in its interior, I, by:

$$A = I + \frac{B}{2} - 1.$$

Substituting $A = 18$ and $B = 4$ gives

$$18 = I + \frac{4}{2} - 1 \implies I = \boxed{17}.$$

G8. Sophia and Ellie are standing at points $(1, 4)$ and $(8, 12)$, respectively, in the Cartesian plane. There is a wall along the line $y = \frac{x}{2} + 1$. Sophia throws a ball so that it travels in a straight line, bounces off the wall and then continues in a straight line to Ellie. Find the distance that the ball travels.

Proposed by: Paige Zhu

Answer: $\boxed{10}$

Solution: Observe that we can reflect Sophia's position over the line representing the wall, and then the length of the path the ball follows is the same as the straight line distance from Sophia's reflected location to Ellie's location. From graphing, we can see that the reflection of $(1, 4)$ over the line $y = \frac{x}{2} + 1$ is just $(3, 0)$. This point can also be found by finding the intersection of the line perpendicular to the wall passing through $(1, 4)$ with the wall, and then reflecting $(1, 4)$ over this intersection point. Then, the distance from $(3, 0)$ to $(8, 12)$ is just

$$\sqrt{(8-3)^2 + (12-0)^2} = \boxed{13}.$$

G9. Satabhisha is standing at the origin of the Cartesian plane. Every minute, she picks a random direction to walk, either up, down, left, or right, and moves one unit in that direction. After five minutes she is on (x, y). If the probability that $|x| + |y| = 5$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime integers, find $m + n$.

Proposed by: Paige Zhu

Answer: $\boxed{287}$

Solution: In order for Satabhisha's ending location to be valid, she cannot move both up and down, or both left and right within the 5 minute span. To count the number of valid paths, we can choose either of the vertical directions that she must stick to (if she decides to move vertically), and either of the horizontal directions that she must stick to (if she decides to move horizontally). This can be done in $2 \cdot 2 = 4$ ways. Without loss of generality, suppose she only moves either up or right. The number of 5 step paths moving either up or right is $2^5 = 32$. However, we have to be careful to not overcount the cases where she only moves in one direction for all five steps, so we must subtract 4 paths from our total count. Thus, the answer is

$$\frac{4 \cdot 32 - 4}{4^5} = \frac{31}{256} \implies m + n = \boxed{287}.$$

G10. The minimum value of the expression

$$\frac{x^3}{y} + \frac{y^3}{z} + \frac{z^3}{x},$$

for positive real numbers x, y, z where $x + y + z = 1$, can be written as a fraction $\frac{p}{q}$, where p and q are relatively prime integers. Find $p + q$.

Proposed by: Paige Zhu

Answer: $\boxed{4}$

Solution 1: Let S denote the given expression in the problem. Apply the Cauchy-Schwarz inequality as follows:

$$S(xy + yz + xz) \geq (x^2 + y^2 + z^2)^2.$$

The Cauchy-Schwarz inequality also gives

$$\left(x^2 + y^2 + z^2\right)(1 + 1 + 1) \geq (x + y + z)^2 = 1 \implies x^2 + y^2 + z^2 \geq \frac{1}{3}.$$

We also know that

$$1 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx).$$

Rearranging gives

$$xy + yz + zx = \frac{1 - \left(x^2 + y^2 + z^2\right)}{2} \implies xy + yz + zx \leq \frac{1}{3}.$$

Then, from the first inequality, we have that

$$S \geq \frac{(x^2 + y^2 + z^2)^2}{xy + yz + zx} \geq \frac{\left(\frac{1}{3}\right)^2}{\frac{1}{3}} = \frac{1}{3}.$$

Indeed, this value is achieved by taking $x = y = z = \frac{1}{3}$. Hence, the answer is $1 + 3 = \boxed{4}$.

Solution 2: Alternatively, if the reader is unfamiliar with the Cauchy-Schwarz inequality, we present a solution using only the AM-GM inequality. Again, let S denote the given expression in the problem. From AM-GM, we have that

$$\frac{x^3}{y} + xy \geq 2\sqrt{\frac{x^3}{y} \cdot xy} = 2x^2.$$

Similarly, we have

$$\frac{y^3}{z} + yz \geq 2y^2, \qquad \frac{z^3}{x} + zx \geq 2z^2.$$

Then, we have

$$S + (xy + yz + zx) \geq 2(x^2 + y^2 + z^2).$$

From AM-GM, we have that

$$x^2 + y^2 \geq 2\sqrt{x^2y^2} = 2xy, \quad y^2 + z^2 \geq 2yz, \quad z^2 + x^2 \geq 2zx.$$

Adding and dividing by 2 gives

$$x^2 + y^2 + z^2 \geq xy + yz + zx.$$

Hence, we have $S \geq 2(x^2 + y^2 + z^2) - (xy + yz + zx) \geq x^2 + y^2 + z^2$. Using the given condition $x + y + z = 1$, we have:

$$1 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) \leq 3(x^2 + y^2 + z^2).$$

Hence, we have $S \geq x^2 + y^2 + z^2 \geq \frac{1}{3}$. This is achieved when $x = y = z = \frac{1}{3}$, so the answer is $1 + 3 = \boxed{4}$.

G11. Find the smallest positive integer k such that

$$1^{2025} + 2^{2025} + \ldots + k^{2025}$$

is divisible by 2025.

Proposed by: Brandon Xu

Answer: $\boxed{5}$

Solution 1: Let S denote the given sum. Consider mod 25 and mod 81 separately.

For any integer a relatively prime to 25, we have by Euler totient's theorem that $a^{\varphi(25)} = a^{20} \equiv 1 \pmod{25}$. Then, we have

$$S \equiv \sum_{n=0}^{k} n^5 \pmod{25}.$$

Writing $n = 5p + q$, the Binomial Theorem gives

$$n^5 = (5p + q)^5 = \sum_{i=0}^{5} \binom{5}{i} (5p)^i q^{5-i} \equiv q^5 \pmod{25}.$$

Hence, it suffices to check

$$S_k \pmod{25}$$

for $k = 0, 1, 2, 3, 4$. We have

$$\begin{aligned} S_0 &\equiv 0 \pmod{25} \\ S_1 &\equiv 1 \pmod{25} \\ S_2 &\equiv 8 \pmod{25} \\ S_3 &\equiv 1 \pmod{25} \\ S_4 &\equiv 0 \pmod{25} \\ S_5 &\equiv 0 \pmod{25} \\ &\vdots \end{aligned}$$

Similarly, for mod 81, we have by Euler totient's theorem that $a^{\varphi(81)} = a^{54} \equiv 1 \pmod{81}$ for any integer a relatively prime to 81. Since $2025 \equiv 27 \pmod{54}$, we have

$$S \equiv \sum_{n=0}^{k} n^{27} \pmod{81}.$$

Notice that for any integer n relatively prime to 81, we have

$$\left(n^{27}\right)^2 = n^{54} \equiv 1 \pmod{81} \implies n^{27} = \pm 1 \pmod{81}.$$

Writing $n = 3p + q$, we have that

$$n^{27} = (3p+q)^{27} \equiv q^{27} \pmod{81}$$

by the Binomial theorem. Hence, we have that:

$$\begin{aligned} S_0 &\equiv 0 \pmod{81} \\ S_1 &\equiv 1 \pmod{81} \\ S_2 &\equiv 0 \pmod{81} \\ S_3 &\equiv 0 \pmod{81} \\ &\vdots \end{aligned}$$

The smallest k such that $S_k \equiv 0 \pmod{25}$ and $S_k \equiv 0 \pmod{81}$ is just $\boxed{5}$.

G12. Triangle ABC is inscribed in circle ω. Let M be the midpoint of side AC, and let the median BM be extended to intersect ω again at E. Given that $AB = 5, BC = 3$, and $AC = 6$, the value of ME can be written as $\frac{p}{q}\sqrt{r}$ where p and r are relatively prime integers, and q is not divisible by the square of any prime. Find $p + q + r$.

Proposed by: Paige Zhu

Answer: $\boxed{15}$

Solution: Let M be the midpoint of AC, so $MA = MC = 3$. Since A, B, C, E lie on the same circle, the power of point M gives $MA \cdot MC = MB \cdot ME$. Thus

$$MB \cdot ME = 3 \cdot 3 = 9,$$

so

$$ME = \frac{9}{MB}.$$

We compute MB using the median length formula. The median from vertex B to side AC has length

$$MB = \frac{1}{2}\sqrt{2AB^2 + 2BC^2 - AC^2}.$$

Substituting the given side lengths yields

$$MB = \frac{1}{2}\sqrt{2 \cdot 5^2 + 2 \cdot 3^2 - 6^2} = \frac{1}{2}\sqrt{50 + 18 - 36} = \frac{1}{2}\sqrt{32} = 2\sqrt{2}.$$

Therefore,

$$ME = \frac{9}{2\sqrt{2}} = \frac{9\sqrt{2}}{4}.$$

Write $ME = \dfrac{p}{q}\sqrt{r}$, so $p = 9$, $q = 4$, and $r = 2$. Hence $p + q + r = 9 + 4 + 2 = \boxed{15}$.

Note: the length of segment MB can also be computed using Stewart's theorem.

G13. Carolyn the frog is hopping along the number line. She starts at the number 0, and stops hopping once she reaches the number 100. If her current position is the integer k, she will randomly hop to a new integer position between $k+1$ and 100, inclusive. Given the probability that Carolyn lands on 75 at some point can be expressed as $\frac{m}{n}$ where m and n are relatively prime integers, find $m+n$.

Proposed by: Carolyn Cao

Answer: $\boxed{27}$

Solution: Let p_k denote the probability that Carolyn ever lands on 75 when she starts at integer k. If $k \geq 76$ then $p_k = 0$, and $p_{75} = 1$. On the other hand, if $k \leq 74$ then the rule gives

$$p_k = \frac{1}{100-k} \sum_{j=k+1}^{100} p_j.$$

We prove by backward induction that $p_k = \dfrac{1}{26}$ for every $k \leq 74$. For the base case $k = 74$ there are $100 - 74 = 26$ possible hops, and exactly one of those lands at 75, so

$$p_{74} = \frac{1}{26}.$$

Now assume $p_t = \dfrac{1}{26}$ for all integers t with $k < t \leq 74$. Then

$$p_k = \frac{1}{100-k}\left(\sum_{j=k+1}^{74} \frac{1}{26} + p_{75}\right) = \frac{1}{100-k}\left(\frac{74-k}{26} + 1\right).$$

Combine terms in the numerator to get

$$p_k = \frac{1}{100-k} \cdot \frac{100-k}{26} = \frac{1}{26}.$$

Thus the induction step holds, and $p_k = \dfrac{1}{26}$ for every $k \leq 74$.

Hence, the answer is $1 + 26 = \boxed{27}$.

G14. Find the number of quartic polynomials $P(x)$ with integer coefficients satisfying

$$P(1) = 1, P(2) = 4, P(3) = 9, P(4) = 16,$$

and $|P(6)| < 1000$.

Proposed by: Paige Zhu

Answer: $\boxed{16}$

Solution: Let $Q(x) = P(x) - x^2$, then Q is a polynomial with integer coefficients that vanishes at $x = 1, 2, 3, 4$. Hence Q is a multiple of the monic quartic factor $(x-1)(x-2)(x-3)(x-4)$. Write

$$P(x) = x^2 + k(x-1)(x-2)(x-3)(x-4),$$

where k is an integer, and the polynomial is quartic exactly when $k \neq 0$, otherwise the polynomial has degree two.

Compute $P(6)$, then

$$P(6) = 36 + k \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 36 + 120k.$$

The inequality $|P(6)| < 1000$ becomes

$$|36 + 120k| < 1000.$$

Solving the inequality yields

$$-1000 < 36 + 120k < 1000,$$

so

$$-\frac{1036}{120} < k < \frac{964}{120},$$

which gives the integer range $k \in \{-8, -7, \ldots, 7, 8\}$.

This list contains 17 integers, but we must exclude $k = 0$ to ensure that P is quartic. Therefore there are $17 - 1 = \boxed{16}$ admissible polynomials.

G15. Let $ABCD$ be a square such that all four of its vertices lie on the graph of

$$2025y^2 = |2024x + 2025|.$$

If AB can be expressed as $\frac{m}{n}$ where m and n are relatively prime integers, find $m + n$.

Proposed by: Brandon Xu

Answer: $\boxed{6073}$

Solution: Rewrite the given equation as

$$2025y^2 = 2024\left|x + \frac{2025}{2024}\right|.$$

Note that the graph of this equation is just the graph of

$$2025y^2 = 2024|x| \tag{1}$$

shifted 2025/2024 units to the left. So, we can just solve for the side of the square lying on the graph of this shifted equation. Since the graph of (1) is symmetric under reflection over both the x and y axes, and noting the fact that the four points $(a, a), (-a, a), (-a, -a), (a, -a)$ always form a square, we can simply find the intersection of $y = x$ with the graph of (1). Solving yields

$$2025x^2 = 2024x \implies x = 2024/2025.$$

So, the side length of the square is just

$$2 \cdot \frac{2024}{2025} = \frac{4048}{2025}.$$

The answer is $4048 + 2025 = \boxed{6073}$.

G16. Consider the sequence of integers a_n which satisfies $a_1 = 1$, $a_2 = 2$, and

$$a_{k+1} = 2a_k - 23a_{k-1}.$$

Find the remainder when a_{2026} is divided by 2024.

Proposed by: Brandon Xu

Answer: $\boxed{2}$

Solution: We consider mod 8, 11, and 23 separately. Taking the given relation mod 8, we have

$$a_{k+1} \equiv 2a_k + a_{k-1} \pmod 8.$$

After listing the first few terms of a_n mod 8, we can see that $a_9 \equiv 1 \pmod 8$ and $a_{10} \equiv 2 \pmod 8$, and $a_{n+8} \equiv a_n \pmod 8$. Then, we have $a_{2026} \equiv a_2 \equiv 2 \pmod 8$.

Taking the given relation mod 11, we have that

$$a_{k+1} \equiv 2a_k - a_{k-1} \pmod{11},$$

$$a_{k+1} - a_k \equiv a_k - a_{k-1} \pmod{11}.$$

Since $a_2 - a_1 = 1$, we have that $a_k \equiv k \pmod{11}$. For $k = 2026$, we have $a_k \equiv 2026 \equiv 2 \pmod{11}$.

Notice that taking the given relation mod 23, we have that

$$a_{k+1} \equiv 2a_k \pmod{23} \implies a_k \equiv 2^k \pmod{23}.$$

For $k = 2026$, we have $a_k = 2^{2026} \equiv 2^{22\cdot 92+1} \equiv 2 \pmod{23}$, since $2^{22} \equiv 1 \pmod{23}$ by Fermat's Little Theorem.

Hence, a_{2026} is equivalent to 2 mod 8, 11, and 23. Therefore, by the Chinese Remainder Theorem,

$$a_{2026} \equiv \boxed{2} \pmod{2024}.$$

G17. Each point in a 16×16 lattice is independently colored red with probability p or blue with probability $1 - p$. Let X denote the number of ordered quadruples (P_1, P_2, P_3, P_4) of distinct points such that:

- P_1 and P_2 lie in the same column,
- P_2, P_3, P_4 lie in the same row,
- P_1,P_3, P_4 are red, and P_2 is blue.

Compute the maximum expected value of X over all values of p.

Proposed by: Brandon Xu

Answer: $\boxed{85050}$

Solution: We find the probability that any random quadruple of points satisfying the first two conditions also satisfies the third coloring condition. Since the coloring of the points is independent, this probability is just $p^3(1 - p)$. Next, we calculate the number of ordered quadruples of points satisfying the first two conditions. We can choose the column shared by P_1 and P_2 in 16 ways, and the row shared by P_2, P_3, P_4 in 16 ways. We can choose the row containing P_1 in 15 ways, and columns for P_3 and P_4 in $15 \cdot 14$ ways. Hence, the total number of quadruples is simply $16^2 \cdot 15^2 \cdot 14$. The expected value of X is simply

$$\mathbb{E}(X) = 16^2 \cdot 15^2 \cdot 14 \cdot p^3(1 - p),$$

by linearity of expectation. Thus, to maximize $\mathbb{E}(X)$, we must maximize $p^3(1-p)$. By the AM-GM inequality, we have

$$1 = \frac{p}{3} + \frac{p}{3} + \frac{p}{3} + (1-p) \geq 4\left(\frac{p^3(1-p)}{27}\right)^{\frac{1}{4}} \implies p^3(1-p) \leq \frac{27}{256}$$

achieved when $p = \frac{3}{4}$. Hence, the answer is

$$16^2 \cdot 15^2 \cdot 14 \cdot \frac{27}{256} = \boxed{85050}.$$

G18. A polynomial $p(x)$ with real coefficients satisfies

$$p(p(x)) = 7p(x) - p(-7x) + 2025.$$

Find the sum of all possible values of $|p(1)|$.

Proposed by: Oliver Zhang

Answer: $\boxed{644}$

Solution: Suppose deg $(p) = d$. We have

$$d^2 = d \implies d = 0, 1.$$

Therefore, $p(x) = c$ or $p(x) = ax + b$.

a) Case 1: $p(x) = c$

In this case, we have $c = 7c - c + 2025 \implies -5c = 2025 \implies c = -405$, so $p(x) = -405$ and $|p(1)| = 405$.

b) Case 2: $p(x) = ax + b$

In this case, we expand both sides of the original equation:

$$\begin{aligned} a(ax+b) + b &= 7(ax+b) - (-7ax+b) + 2025 \\ a^2x + (ab+b) &= 14ax + 6b + 2025 \\ a^2x + ab &= 14ax + 5b + 2025. \end{aligned}$$

Therefore,

$$\begin{cases} a^2 = 14a \\ ab = 5b + 2025 \end{cases} \implies \begin{cases} a = 14 \\ 14b + 5b + 2025 \implies b = 225. \end{cases}$$

So $p(x) = 14x + 225$. Then $|p(1)| = 239$.

Finally, $405 + 239 = \boxed{644}$.

G19. Suppose $x, y, z > 0$ are real numbers that satisfy

$$\begin{aligned} \frac{x+1}{1+y} + \frac{y+1}{1+z} + \frac{z+1}{1+x} &= \frac{10}{3} \\ \frac{x+1}{1+z} + \frac{y+1}{1+x} + \frac{z+1}{1+y} &= \frac{41}{12} \\ \frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} &= \frac{9}{5}. \end{aligned}$$

If $x+y+z$ can be written as $\frac{m}{n}$ where m and n are relatively prime integers, find $m+n$.

Proposed by: Jialai She

Answer: $\boxed{41}$

Solution: Let $A = \frac{1}{1+x}$, $B = \frac{1}{1+y}$, and $C = \frac{1}{1+z}$. We wish to find $x+y+z = \frac{1}{A} + \frac{1}{B} + \frac{1}{C} - 3$.

We can rewrite the system of equations as

$$\frac{B}{A} + \frac{C}{B} + \frac{A}{C} = \frac{10}{3}$$
$$\frac{C}{A} + \frac{A}{B} + \frac{B}{C} = \frac{41}{12}$$
$$A + B + C = \frac{9}{5}.$$

Add the first two equations to get

$$\frac{(B+C)}{A} + \frac{(A+C)}{B} + \frac{(A+B)}{C} = \frac{27}{4}. \tag{2}$$

We can use the third equation to rewrite Equation (2):

$$\frac{9/5 - A}{A} + \frac{9/5 - B}{B} + \frac{9/5 - C}{C} = \frac{27}{4}$$

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} = \frac{5}{9} \cdot \left(\frac{27}{4} + 3\right) = \frac{65}{12}.$$

Hence, we have

$$x+y+z = \frac{65}{12} - 3 = \frac{29}{12} \implies m+n = \boxed{41}.$$

G20. Let S be a (possibly empty) subset of the integers from 1 to 14, inclusive. Oliver is walking along the number line, starting from the origin. At each moment, he moves 1 or 2 units in the positive direction. However, he does not want to visit any number that is in S. For how many sets S is it possible for Oliver to reach the point 15?

Proposed by: Oliver Zhang

Answer: $\boxed{987}$

Solution: We claim in general that the number of subsets S on the integers from 1 to $k-1$ such that it is possible for Oliver to reach point k without landing on a number in S is exactly the $k+1$-st Fibonacci number, F_{k+1}.

Let S_k denote the number of such subsets for which a valid path from 1 to k is possible. Consider the number $k-1$. If $k-1 \in S$, any valid path to k must land on $k-2$, and $k-2$ cannot be in S. Hence, the number of valid S in this case must be S_{k-2}. Otherwise, if $k-1 \notin S$, then any subset S counted in S_{k_1} is still valid. So, we have the recurrence $S_k = S_{k-1} + S_{k-2}$. Clearly, we have $S_1 = 1 = F_2$ (the empty set), and $S_2 = 2 = F_3$ (either S is empty, or $S = \{1\}$). So, in general, we have $S_k = F_{k+1}$. After some quick computation, we have $S_{15} = F_{16} = \boxed{987}$.

G21. How many 6-digit positive integers, containing only the digits 1, 2, 3, and 4, have the property that the sum of the first two digits is less than the sum of the next two digits, which is also less than the sum of the last two digits?

Proposed by: Brandon Xu

Answer: $\boxed{376}$

Solution: We must count 6-digit integers of the form

$$\overline{d_1d_2d_3d_4d_5d_6},$$

where each $d_i \in \{1,2,3,4\}$, that satisfy

$$(d_1 + d_2) < (d_3 + d_4) < (d_5 + d_6).$$

Let the three pairwise sums be

$$s_1 = d_1 + d_2, \quad s_2 = d_3 + d_4, \quad s_3 = d_5 + d_6.$$

Each d_i is between 1 and 4, so each sum s_j lies between 2 and 8.

First, count the number of ordered pairs (a, b) with $a, b \in \{1,2,3,4\}$ that yield a given sum $s = a + b$.

We have:

s	$\#\{(a,b) \in \{1,2,3,4\}^2 : a+b=s\}$
2	$(1,1) \Rightarrow 1$
3	$(1,2),(2,1) \Rightarrow 2$
4	$(1,3),(3,1),(2,2) \Rightarrow 3$
5	$(1,4),(4,1),(2,3),(3,2) \Rightarrow 4$
6	$(2,4),(4,2),(3,3) \Rightarrow 3$
7	$(3,4),(4,3) \Rightarrow 2$
8	$(4,4) \Rightarrow 1$

Define $f(s)$ to be the number of ordered pairs with sum s. Then

$$f(2), f(3), f(4), f(5), f(6), f(7), f(8) = 1, 2, 3, 4, 3, 2, 1.$$

Now, for each choice of sums $s_1 < s_2 < s_3$ with $s_j \in \{2, 3, \ldots, 8\}$, the number of 6-digit numbers that realize these sums is

$$f(s_1) \cdot f(s_2) \cdot f(s_3),$$

since the choices for each pair (d_1, d_2), (d_3, d_4), (d_5, d_6) are independent once the sums are fixed.

Therefore, the desired total is

$$\sum_{2 \le s_1 < s_2 < s_3 \le 8} f(s_1) f(s_2) f(s_3).$$

Let

$$T = \sum_{s=2}^{8} f(s) = 16, \qquad U = \sum_{s=2}^{8} f(s)^2 = 44, \qquad V = \sum_{s=2}^{8} f(s)^3 = 136.$$

Consider the sum over all ordered triples:

$$\sum_{i,j,k} f(i)f(j)f(k) = T^3.$$

We subtract the triples in which indices are equal. The triples s_1, s_2, s_3 with exactly two equal s_i, s_j contribute

$$3\sum_i f(i)^2(T - f(i)) = 3(TU - V),$$

and triples with all three equal contribute V. Thus the sum over distinct s_1, s_2, s_3 is

$$S_{\text{distinct}} = T^3 - 3(TU - V) - V = 4096 - 2112 + 272 = 2256.$$

Each unordered triple $\{s_1, s_2, s_3\}$ with $s_1 < s_2 < s_3$ corresponds to 6 ordered triples, so the answer is

$$\frac{S_{\text{distinct}}}{6} = \frac{2256}{6} = \boxed{376}.$$

G22. Let n and k be positive integers. Define $G(n, k)$ as the number of functions f from $\{0, 1, 2, \ldots, n\}$ to the integers such that $f(0) = 0$, $f(n) = n(k - 2)$, and

$$|f(x) - f(x - 1)| \le k$$

for all $x \in \{1, 2, \ldots, n\}$. Given that $G(n, k) > 2025$, find the smallest possible value of kn.

Proposed by: Brandon Xu

Answer: $\boxed{14}$

Solution: Let $d_x = f(x) - f(x - 1) \in \mathbb{Z}$ for $1 \le x \le n$. Then

$$|d_x| \le k, \qquad \sum_{x=1}^{n} d_x = f(n) - f(0) = n(k - 2).$$

Setting $e_x = k - d_x$. Then $0 \le e_x \le 2k$ and

$$\sum_{x=1}^{n} e_x = \sum_{x=1}^{n}(k - d_x) = nk - n(k - 2) = 2n.$$

So, $G(n, k)$ equals the number of n-tuples $(e_1, \ldots, e_n)$ with

$$e_1 + \cdots + e_n = 2n, \qquad 0 \le e_i \le 2k.$$

By inclusion-exclusion, with $S = 2n$ and upper bound $2k$,

$$G(n, k) = \sum_{j=0}^{\lfloor S/(2k+1) \rfloor} (-1)^j \binom{n}{j} \binom{S - j(2k + 1) + n - 1}{n - 1}.$$

If $k = 1$, then $|d_x| \le 1$ forces $\sum d_x \in [-n, n]$, but $n(k-2) = -n$, so only the unique choice $d_x = -1$ works; hence $G(n, 1) = 1$.

So consider $k = 2$. Some trial and error gives: For $n = 6$,

$$G(6, 2) = \binom{17}{5} - 6\binom{12}{5} + 15\binom{7}{5} = 1751 < 2025.$$

For $n = 7$,

$$G(7,2) = \binom{20}{6} - 7\binom{15}{6} + 21\binom{10}{6} = 38760 - 35035 + 4410 = 8135 > 2025.$$

If $k \geq 3$ and $kn \leq 13$, then $n \leq 4$. Dropping the upper bounds $e_i \leq 2k$ can only increase the count, so

$$G(n,k) \leq \binom{2n+n-1}{n-1} = \binom{3n-1}{n-1} \leq \binom{11}{3} < 2025.$$

Hence the smallest n with $G(n,2) > 2025$ is $n = 7$, giving the smallest possible product

$$kn = 2 \cdot 7 = \boxed{14}.$$

G23. Let $ABCDEF$ be a hexagon inscribed in a circle ω. Lines AD and CE intersect at P, lines CF and AE intersect at Q, and lines EB and AC intersect at R. Given that $AC = 8$, $CE = 6$, and $EA = 10$, and that $ABCDEF$ is the unique hexagon which minimizes

$$\frac{AD}{DP} + \frac{CF}{FQ} + \frac{EB}{BR},$$

find the area of $ABCDEF$.

Proposed by: Brandon Xu

Answer: $\boxed{60}$

Solution: Let the foot of the altitude from C to AE be X, and the foot of the altitude from D to CE be X'. By similar triangles, we have

$$\frac{AP}{PD} = \frac{AX}{DX'} \implies \frac{AD}{DP} = 1 + \frac{AX}{DX'}.$$

Since AX is fixed, the optimal point D minimizing $\frac{AD}{DP}$ is simply the midpoint of the minor arc CE, or that AD bisects $\angle EAC$. Similarly, we can see that EB is the angle bisector of $\angle CEA$, and FC is the angle bisector of $\angle ACE$. We can simply compute the area

$$[ABCDEF] = [ABC] + [CDE] + [EFA] + [ACE].$$

Let M be the midpoint of AE. Note that $MF = 5$ since M must be the center of ω. Also, since M bisects AC and CE, we know that the distance from B to AC is 2 and the distance from D to CE is 1. Hence, the total area is just

$$[ABCDEF] = \frac{1}{2} \cdot 8 \cdot 2 + \frac{1}{2} \cdot 6 \cdot 1 + \frac{1}{2} \cdot 10 \cdot 5 + \frac{1}{2} \cdot 6 \cdot 8 = \boxed{60}.$$

G24. Compute the number of positive integers x and y satisfying

$$2x^3 \equiv y^4 \pmod{2025}$$

where $1 \leq x \leq 2025$ and $1 \leq y \leq 2025$.

Proposed by: Brandon Xu

Answer: $\boxed{13365}$

Solution: Let $2025 = 3^4 \cdot 5^2 = 81 \cdot 25$. Since $\gcd(81, 25) = 1$, by CRT the number of pairs $(x, y) \pmod{2025}$ with

$$2x^3 \equiv y^4 \pmod{2025}$$

equals $N_{81}N_{25}$, where N_m counts solutions mod m.

If $9 \mid x$, then $81 \mid 2x^3$, so we need $81 \mid y^4$, equivalently $3 \mid y$. There are $81/9 = 9$ choices for $x \equiv 0 \pmod 9$ and $81/3 = 27$ choices for $y \equiv 0 \pmod 3$, giving 243 solutions.

If $3 \nmid x$, then $2x^3$ is a unit mod 81, so y must also be a unit, hence $3 \nmid y$. Let $\varphi(81) = 54$ and fix a primitive root g mod 81. Write $2 \equiv g^c$, $x \equiv g^a$, $y \equiv g^b$ with $a, b, c \in \{0, 1, \ldots, 53\}$. Then

$$2x^3 \equiv y^4 \iff c + 3a \equiv 4b \pmod{54}.$$

For a given a, this has a solution b iff $c + 3a \equiv 0 \pmod 2$, i.e. iff $c + a$ is even (since $3a \equiv a \pmod 2$), and then there are exactly 2 solutions for b modulo 54. As a ranges over $0, \ldots, 53$, exactly 27 values make $c + a$ even, so the unit solutions contribute $27 \cdot 2 = 54$.

Thus

$$N_{81} = 243 + 54 = 297.$$

If $5 \mid x$, then $25 \mid 2x^3$, so we need $25 \mid y^4$, equivalently $5 \mid y$. There are $25/5 = 5$ choices for each of $x \equiv 0 \pmod 5$ and $y \equiv 0 \pmod 5$, giving 25 solutions.

If $5 \nmid x$, then $5 \nmid y$. Let $\varphi(25) = 20$ and fix a primitive root g mod 25. Write $2 \equiv g^c$, $x \equiv g^a$, $y \equiv g^b$ with $a, b, c \in \{0, 1, \ldots, 19\}$. Then

$$2x^3 \equiv y^4 \iff c + 3a \equiv 4b \pmod{20}.$$

Since $\gcd(3, 20) = 1$, as a varies, $c + 3a$ runs through all residues mod 20, so exactly $1/4$ of the 20 values satisfy $c + 3a \equiv 0 \pmod 4$, namely 5 values of a. For each such a, the congruence $4b \equiv c + 3a \pmod{20}$ has exactly 4 solutions b (because $\gcd(4, 20) = 4$). Hence the unit solutions contribute $5 \cdot 4 = 20$.

Thus

$$N_{25} = 25 + 20 = 45.$$

Therefore the total number of solutions with $1 \le x, y \le 2025$ is

$$N_{2025} = N_{81}N_{25} = 297 \cdot 45 = \boxed{13365}.$$