# MOAA 2025 Speed Round Solutions

Math Open at Andover

October 11th, 2025

S1. Compute $20 \times 25 + \sqrt{2025} + 2025$.

*Proposed by: Paige Zhu*

**Answer:** $\boxed{2570}$

**Solution:** Note that $\sqrt{2025} = 45$. Straightforward calculation gives

$$20 \times 25 + \sqrt{2025} + 2025 = 550 + 45 + 2025 = \boxed{2570}.$$

S2. Let $ABCD$ be a square. Let $M$ be the midpoint of $CD$, and $N$ be the midpoint of $AM$. Given that the area of triangle $BMN$ is 2025, find the length of $AB$.

*Proposed by: Brandon Xu*

**Answer:** $\boxed{90}$

**Solution:** Let $s = AB$. Since $AN = MN$, we must have that $[BAN] = [BMN]$. Solving for the area of $[BAN]$ in terms of $s$ gives:

$$[BAN] = \frac{1}{2} \cdot s \cdot \frac{s}{2} = \frac{s^2}{4} = 2025 \implies s = \boxed{90}.$$

S3. Find the number of pairs $(m, n)$ such that $m + n = 2025$ and $\gcd(m, n) = 1$.

*Proposed by: Paige Zhu*

**Answer:** $\boxed{1080}$

**Solution:** Recall the fact that $\gcd(a, b) = \gcd(a - b, b)$. We have

$$\gcd(m, n) = \gcd(m, 2025 - m) = \gcd(m, 2025) = 1.$$

Hence, the pair $(m, n)$ is valid if and only if $\gcd(m, 2025) = 1$. Since $2025 = 3^4 \cdot 5^2$, $m$ must not be divisible by 3 or 5. Hence, our answer is

$$\varphi(2025) = 2025\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) = \boxed{1080}.$$

Alternatively, if the reader is unfamiliar with the Euler totient function $\varphi(n)$, one can notice that if $n$ is not divisible by 3 or 5, then $n + 15$ is not divisible by 3 or 5. Then, since there are 8 integers between 0 and 14, inclusive, which are not divisible by 3 or 5, we can simply compute

$$2025 \cdot \frac{8}{15} = \boxed{1080}$$

as before.

S4. At Areteem, Eddie and Eugenia simultaneously begin reading the same book. Eddie reads 12 pages on day 1, and on each subsequent day he reads 3 more pages than the previous day. Eugenia reads 3 pages on day 1, and on each subsequent day she reads 5 more pages than the previous day. At the end of day $M$, they both have 1 page left to read. If the book contains $N$ pages, find $N + M$.

*Proposed by: Brandon Xu*

**Answer:** $\boxed{266}$

**Solution:** The number of pages Eddie reads by the end of day $M$ is:

$$12 + 15 + \cdots + [12 + 3(M - 1)] = \frac{M}{2}[24 + 3(M - 1)] = \frac{M}{2}(21 + 3M).$$

The number of pages Eugenia reads by the end of day $M$ is:

$$3 + 8 + \cdots + [3 + 5(M - 1)] = \frac{M}{2}[6 + 5(M - 1)] = \frac{M}{2}(1 + 5M).$$

Hence, we must have

$$\frac{M}{2}(21 + 3M) = \frac{M}{2}(1 + 5M) \implies 21 + 3M = 1 + 5M,$$

so $M = 10$. Then, we have $N = \frac{M}{2}(1 + 5M) + 1 = 5 \cdot 51 + 1 = 256$. Hence, $N + M = \boxed{266}$.

S5. Let $S$ be the set of all positive integers $n$ such that $n$ and $n^2$ both end in the same three-digit number $\underline{a}\,\underline{b}\,\underline{c}$, with $a > 0$. Compute the fifth smallest number in $S$.

*Proposed by: Paige Zhu*

**Answer:** $\boxed{2376}$

**Solution:** We want all positive integers $n$ such that $n$ and $n^2$ end in the same three digits $abc$ where $a > 0$. This means that $n^2 \equiv n \pmod{1000}$, or equivalently $n(n-1) \equiv 0 \pmod{1000}$.

Since $1000 = 2^3 \cdot 5^3$, we need $n(n-1)$ to be divisible by both 8 and 125. Because $n$ and $n - 1$ are consecutive integers, one of them must be divisible by 8, and one of them must be divisible by 125. Thus, $n \equiv 0$ or $1 \pmod{8}$, and also $n \equiv 0$ or $1 \pmod{125}$. By the Chinese Remainder Theorem, there are $2 \times 2 = 4$ possible solutions modulo 1000:

1. $n \equiv 0 \pmod{8}$ and $n \equiv 0 \pmod{125}$ gives $n \equiv 0 \pmod{1000}$.
2. $n \equiv 1 \pmod{8}$ and $n \equiv 1 \pmod{125}$ gives $n \equiv 1 \pmod{1000}$.
3. $n \equiv 0 \pmod{8}$ and $n \equiv 1 \pmod{125}$ gives $n \equiv 376 \pmod{1000}$.
4. $n \equiv 1 \pmod{8}$ and $n \equiv 0 \pmod{125}$ gives $n \equiv 625 \pmod{1000}$.

The first two cases give endings 000 and 001, which are not three-digit numbers with $a > 0$, so we ignore them. Hence the valid endings are 376 and 625. Therefore, all such numbers are of the form $n = 1000k + 376$ or $n = 1000k + 625$ for integers $k \geq 0$. Listing them in order, we get $376, 625, 1376, 1625, \boxed{2376}$.

S6. In regular heptagon $\mathcal{H}$, three pairs of vertices are randomly selected, and a line is drawn between every pair. The probability that the three lines drawn bound a triangle with a positive finite area can be written as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m + n$.

*Note: the same pair of vertices can be selected multiple times. For example, one could draw lines $AB$, $AB$, and $AC$.*

*Proposed by: Oliver Zhang*

**Answer:** $\boxed{671}$

**Solution:** Note that a non-degenerate triangle is formed if and only if the three lines are not pairwise parallel, and are not concurrent. Suppose that we select the pairs in order. There are $\binom{7}{2} = 21$ ways to select each pair, so there are $21^3$ ways to select the three pairs.

For every edge of $\mathcal{H}$, there are exactly two other lines parallel to it. Hence, the number of ways to select 3 non-parallel lines is $7 \cdot 6 \cdot 5 \cdot 3^3$, since there are $7 \cdot 6 \cdot 5$ ways to choose 3 distinct directions, then $3^3$ ways to select the specific lines within each direction.

Now, we need to subtract all the sets of lines which are non-parallel, but pass through the same point. The number of ways to choose three concurrent, but not parallel lines is given by $7 \cdot (6 \cdot 5 \cdot 4)$, as there are 7 vertices to choose from, then selecting three of the six lines from that vertex in order. Hence, the desired probability is

$$\frac{7 \cdot 6 \cdot 5 \cdot 3^3}{21^3} - \frac{7 \cdot 6 \cdot 5 \cdot 4}{21^3} = \frac{270}{21^2} - \frac{40}{21^2} = \frac{230}{441}.$$

The answer is $230 + 441 = \boxed{671}$.

S7. Square $ABCD$ has side length 45. Let $E$ be the midpoint of $AB$ and let $F$ be the midpoint of $EB$. Lines $CE$ and $CF$ intersect line $BD$ at $G$ and $H$, respectively. Let $P$ be a point on line $DA$. Lines $CE$ and $CF$ intersect line $BP$ at $X$ and $Y$, respectively. Suppose $XY = 2YB$. Find the area of triangle $PGH$.

*Proposed by: Oliver Zhang*

**Answer:** $\boxed{270}$

**Solution:** Construct a line parallel to $\overline{AB}$ passing through $X$ intersecting lines $CB$ and $CF$ at $Q$ and $R$, respectively. Notice that $\triangle BYF \sim \triangle XYR$, so $XR = 2BF = 2EF$. Also, $\triangle CEF \sim \triangle CXR$ with ratio $CB : CQ$, and therefore $XR = 2EF \implies CQ = 2CB \implies CB = QB$, implying $P$ is the intersection of lines $DA$ and $CE$.

Now, $\triangle CGD \sim \triangle EGB$, so $DG : GB = CD : EB = 2 : 1$. Note that $BCDP$ is a trapezoid, so $[PGB] = [CDG] = \frac{1}{2} \cdot 45^2 \cdot \frac{2}{3} = 675$. Also, $\triangle CHD \sim \triangle FHB$, so $DH : HB = CD : BF = 4 : 1$. From here we see $GH : HB = 2 : 3$. Finally, $[PGH] : [PHB] = GH : HB = 2 : 3$, so $[PGH] = 675 \cdot \frac{2}{5} = \boxed{270}$.

S8. Find the number of ways to label the squares in a $6 \times 6$ grid satisfying the following conditions:

a) Each square contains exactly one number, which is in the set $\{1, 2, 3, 6\}$.

b) The numbers within each row and column multiply to 6.

c) Each row and column contains no more than two numbers greater than 1.

*Proposed by: Brandon Xu*

**Answer:** $\boxed{518400}$

**Solution:** We solve the problem for a general $n \times n$ grid. Let $A$ be any grid satisfying these conditions. Clearly, in any row or column, the entries not equal to

1 must consist of either one 2 and one 3, or a single 6. Let $G_{ij}$ denote the number in the square located in the $i^{\text{th}}$ row and the $j^{\text{th}}$ column of any grid $G$. We fill in each square in $n \times n$ grids $B$ and $C$ as follows:

a) If $A_{ij} = 1$, then $B_{ij} = 1$ and $C_{ij} = 1$.
b) If $A_{ij} = 2$, then $B_{ij} = 2$ and $C_{ij} = 1$.
c) If $A_{ij} = 3$, then $B_{ij} = 1$ and $C_{ij} = 3$.
d) If $A_{ij} = 6$, thhen $B_{ij} = 2$, and $C_{ij} = 3$.

In particular, note that if we multiply the corresponding entries of $B$ and $C$, we return the original grid $A$. Similarly, each grid $A$ can be uniquely decomposed into grids $B$ and $C$. Hence, it suffices to count how many ways to create valid $B$ and $C$.

The entries in $B$ are either 1 or 2, and the entries in $C$ are either 1 or 3. We can also see that each row and column in $B$ has exactly one 2, and each row and column in $C$ has exactly one 3. Consider grid $B$. In the first column, there are $n$ ways to choose where the 2 goes. In the second column, there are $n-1$ ways, since it cannot lie in the same row as the first 2. Similarly, there are $n-2$ ways to place the 2 in the third column, and so on. Hence, there are $n!$ ways to create grid $B$, and similarly, there are $n!$ ways to create grid $C$. Hence, the answer for an $n \times n$ grid is $n!^2$. Taking $n = 6$, we have

$$6!^2 = 720^2 = \boxed{518400}.$$

S9. Find the number of nonnegative real numbers $x$ satisfying the equation

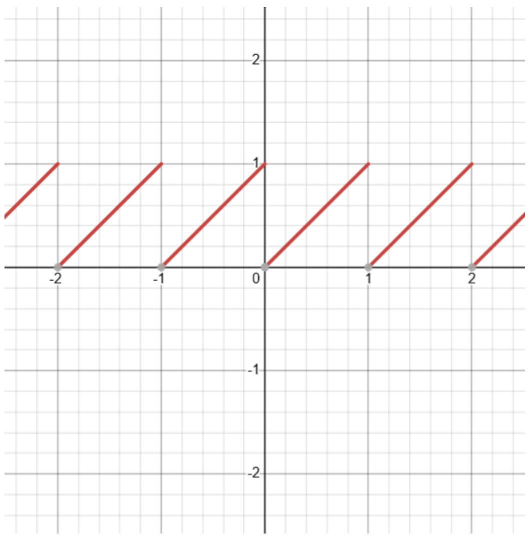
$$\lfloor x \rfloor (x - \lfloor x \rfloor) = \frac{1}{2025} x^2.$$

*Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$.*

*Proposed by: Oliver Zhang*

**Answer:** $\boxed{2023}$

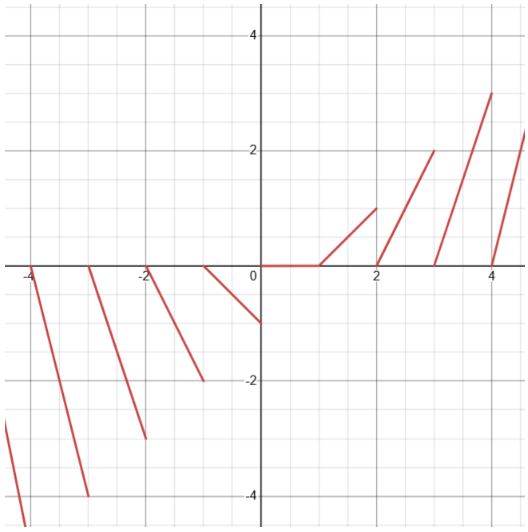
**Solution:** It is easy to visualize the graph of $\{x\}$:




Therefore, the graph of $\lfloor x \rfloor \{x\}$ is just the collection of line segments

$$a(x-a) \quad (a \le x < a+1)$$

for some integer $a$, as shown below.




Notice that for sufficiently large values of $x$, this graph is bounded by the function $y = x - 1$. That is, if $y = \frac{1}{2025}x^2$ intersects $y = x - 1$ at $(x_0, y_0)$, then $\frac{1}{2025}x^2$ must also intersect all line segments $a(x - a)$, where $0 \le a \le (x_0 - 1)$. In other words, if $(x_0, y_0)$ is the solution to $\frac{1}{2025}x^2 = x - 1$, then $\frac{1}{2025}x^2 = \lfloor x \rfloor \{x\}$ has exactly $\lfloor x_0 \rfloor$ nonnegative solutions. Thus, we have

$$\frac{1}{2025}x^2 = x - 1 \implies x^2 - 2025x + 2025 = 0 \implies x = \frac{2025 \pm \sqrt{2025^2 - 4 \cdot 2025}}{2}$$

$$\implies x = \frac{2025 + \sqrt{2025 \cdot 2021}}{2}.$$

It is obvious that $x$ is less than 2025. Also, note that

$$2024 = \frac{2025 + \sqrt{2023 \cdot 2023}}{2},$$

but since $2023 + 2023 = 2025 + 2021$, we have $2023^2 > 2025 \cdot 2021$, so $x < 2024$. Finally,

$$2023 = \frac{2025 + \sqrt{2021 \cdot 2021}}{2},$$

but $\sqrt{2021 \cdot 2021} < \sqrt{2025 \cdot 2021}$, so $\lfloor x \rfloor = 2023$ and the answer is $\boxed{2023}$.

S10. Let $ABC$ be an isosceles triangle with $AB = AC$, where the length of $AB$ is an integer. Let point $P$ lie on side $AB$ and $Q$ on side $AC$ such that $AP = 17$, $AQ = 11$, and $\angle PBC = \angle PQC$. Given the length of $CP$ is a positive integer, find the sum of all the possible lengths of $CP$.

*Proposed by: Brandon Xu*

**Answer:** $\boxed{64}$

**Solution:** Construct the circumcircles of triangles $ABC$ and $APQ$. Extend $PC$ to meet the circumcircle of $ABC$ at point $D$. Note that since $A$, $D$, $B$, $C$ are concyclic,

$$\angle ADC = \angle ABC = 180 - \angle AQP.$$

Thus, $AQPD$ is also cyclic. Then, letting $x$ the length of segment $CQ$, we get from Power of a Point that

$$x(x+11) = CP \cdot CD = CP^2 + CP \cdot PD = CP^2 + 17(x-6).$$

This gives

$$CP^2 = x^2 - 6x + 102 \implies CP^2 = (x-3)^2 + 93.$$

Rearranging and applying difference of squares, we get that

$$93 = CP^2 - (x-3)^2 = (CP - x + 3)(CP + x - 3).$$

Since the length of $CP$ is an integer, we must have either $CP - x + 3 = 1$ and $CP + x - 3 = 93$ which gives $CP = 47$, or $CP - x + 3 = 3$ and $CP + x - 3 = 31$ which gives $CP = 17$. The answer is $47 + 17 = \boxed{64}$.