MOAA 2024 Team Round Solutions

Math Open at Andover

October 5th, 2024

T1. What is the remainder when $2025^{2026^{2027}}$ is divided by 10000?

Proposed by: Paige Zhu

Answer: $\boxed{625}$

Solution: Note that $2025^2 \pmod{10000} \equiv 625$ and $625^2 \pmod{10000} = 625$. Because 2026^{2027} is even, it follows that $2025^{2026^{2027}} \equiv \boxed{625} \pmod{10000}$.

T2. In MOAA-land, each of the three letters M, O, and A cost a whole number of dollars. Given that

- 3 Ms are worth as much as 7 Os
- 8 Os are worth as much as 5 As

What is the minimum number of dollars that the word "MOAA" could cost?

Proposed by: Brandon Xu

Answer: $\boxed{98}$

Solution: Let the costs of the letters M, O, A be x, y, and z dollars, respectively. From the given relations, we find

$$3x = 7y \implies \quad \frac{x}{y} = \frac{7}{3},$$

and

$$8y = 5z \quad \Rightarrow \quad \frac{z}{y} = \frac{8}{5}.$$

Thus, $x = \frac{7}{3}y$ and $z = \frac{8}{5}y$. For both x and z to be integers, y must be a multiple of both 3 and 5. Hence, let $y = 15t$ for some integer $t \geq 1$. Substituting gives

$$M = \frac{7}{3} \cdot 15t = 35t, \qquad A = \frac{8}{5} \cdot 15t = 24t.$$

Therefore, the cost of the word "MOAA" is

$$M + O + A + A = 35t + 15t + 24t + 24t = 98t.$$

The minimum occurs when $t = 1$, so the least possible cost is $\boxed{98}$.

T3. Let p be some number between 0 and 1. If Brandon skips breakfast on a given day, he eats breakfast the next day with probability p. Otherwise, if he eats breakfast on a given day, he eats breakfast again the next day with probability $2p$. Given Brandon ate breakfast on Friday, and the probability that he ate breakfast two days later on Sunday was 72%, find $1000p$, rounded to the nearest integer.

Proposed by: Brandon Xu

Answer: $\boxed{400}$

Solution:

T4. Let $\triangle ABC$ be an equilateral triangle with side length 12. A point P lies strictly inside the triangle such that the distance from P to BC is $\sqrt{3}$, and the areas of triangles PAB and PAC are in the ratio $2:3$. If S is equal to the area of triangle PCA, find S^2.

Proposed by: Paige Zhu

Answer: $\boxed{972}$

Solution: Denote the feet of the perpendiculars from P to AB and AC as D and E, respectively. Set $PD = 2x$. We are given that $PE = 3x$. Solving for the areas of triangles APB, BPC, and CPA and summing them up, we get $6\sqrt{3} + 30x$. We know that the area of an equilateral triangle with side length 12 is $36\sqrt{3}$, so setting our expressions equal gets us $x = \sqrt{3}$.

We plug $x = \sqrt{3}$ back into our equation to get that the area of PCA is $S = \frac{1}{2} \cdot 3\sqrt{3} \cdot 12$, so $S^2 = \boxed{972}$.

T5. Anthony and Gentry are both driving from the same origin to the same destination. Anthony takes a 300-mile route, while Gentry takes a 400-mile route. Their driving speeds (in miles per hour) are chosen independently at random from the interval $[30, 60]$. Given the probability that Anthony arrives at the destination before Gentry can be written as $\frac{p}{q}$, where p and q are relatively prime integers, find $p + q$.

Proposed by: Paige Zhu

Answer: $\boxed{11}$

Solution: Let s_A and s_G be Anthony's and Gentry's speeds, chosen uniformly and independently from the interval $[30, 60]$. Anthony arrives first exactly when

$$\frac{300}{s_A} < \frac{400}{s_G} \quad \Longleftrightarrow \quad s_G < \frac{4}{3} s_A.$$

Represent the sample space by the square $[30, 60] \times [30, 60]$ in the (s_A, s_G)-plane. This square has side length $60 - 30 = 30$, so its total area is $30 \times 30 = 900$. The line $s_G = \frac{4}{3} s_A$ meets the left side $s_A = 30$ at $(30, 40)$ and the top side $s_G = 60$ at $(45, 60)$. Thus, the favorable region $\{(s_A, s_G) : s_G < \frac{4}{3} s_A\}$ inside the square consists of two parts:

- For $45 \le s_A \le 60$, the entire vertical strip $30 \le s_G \le 60$ lies below the line, forming a rectangle of width $60 - 45 = 15$ and height 30. Its area is $15 \times 30 = 450$.
- For $30 \le s_A \le 45$, the portion below the line forms a trapezoid whose vertical heights at $s_A = 30$ and $s_A = 45$ are $40 - 30 = 10$ and $60 - 30 = 30$, respectively. The base length is $45 - 30 = 15$, so its area is $\frac{1}{2}(10 + 30)(15) = 300$.

Thus, the total favorable area is $450 + 300 = 750$. The desired probability is therefore $\frac{750}{900} = \frac{5}{6}$.

Hence, if $\frac{p}{q} = \frac{5}{6}$ in lowest terms, then $p + q = 5 + 6 = \boxed{11}$.

T6. Jonjon has a broken clock which skips over any time showing the number 3. For example, after correctly showing the time 12:02, it will then show the time 12:04 even though it is actually 12:03. If Jonjon accurately sets his clock at 12:00, the time will actually be $a : bc$ when his clock first shows the time 6:00. What is $a + b + c$?

Proposed by: Paige Zhu

Answer: $\boxed{12}$

Solution: We assume time is measured in the 12-hour format, so from 12:00 to 6:00 is a span of 6 hours. We want to know how many *actual minutes* pass before the clock shows 6:00, by only counting times that contain no 3 in either the hour or minute.

We compute how many such valid minutes exist between 12:00 and 6:00. That is, we consider all times from 12:00 to 5:59 inclusive and eliminate those that contain the digit 3. We analyze each hour from 12 to 17.

- Hour 12 is valid (no 3), so we count how many minutes from 00 to 59 do not contain 3.
- Hour 1 is valid.
- Hour 2 is valid.
- Hour 3 contains a 3 $\Rightarrow$ skip.
- Hour 4 is valid.
- Hour 5 is valid.

Thus, we have 5 valid hours: 12, 1, 2, 4, and 5. Now, in each such hour, we want to count how many minutes from 00 to 59 do not contain the digit 3. There are 60 total minutes in an hour. To find how many of them *do not* contain a 3, we subtract the number of minutes that do contain a 3.

From 00 to 59, a minute contains a 3 if either the tens digit or units digit is 3.

Minutes with tens digit 3: 30 to 39, which is 10 values. Minutes with units digit 3: 03, 13, 23, 33, 43, 53, which is 6 values. But 33 is double-counted, so total minutes with a 3:

$$10 + 6 - 1 = 15$$

Thus, each valid hour contributes:

$$60 - 15 = 45 \text{ valid minutes}$$

There are 5 such valid hours, so total valid minutes is:

$$5 \times 45 = 225$$

This means that after 225 real minutes, the broken clock will have displayed 225 times, now showing 6:00. Now convert 225 minutes into hours and minutes:

$$225 \div 60 = 3 \text{ hours and } 45 \text{ minutes}$$

Therefore, the actual time is $3 : 45$, so $a + b + c = \boxed{12}$.

T7. For any positive integer m, let $\tau(m)$ denote the number of divisors of m. How many positive integers n less than 2025 satisfy $\tau(\tau(n)) = 2$ and $\tau(n) \neq 2$?

Proposed by: Paige Zhu

Answer: $\boxed{20}$

Solution: We want to find all positive integers $n < 2025$ such that $\tau(\tau(n)) = 2$ and $\tau(n) \neq 2$. Since $\tau(\tau(n)) = 2$, that means $\tau(n)$ is a prime number. Also, $\tau(n) \neq 2$, so $\tau(n)$ must be an odd prime (3, 5, 7, 11, ...).

If $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, then

$$\tau(n) = (a_1 + 1)(a_2 + 1)\dots(a_k + 1).$$

For this to be prime, the product must have only one factor greater than 1, so n must be a prime power, say $n = p^a$. Then $\tau(n) = a + 1$, and this must be a prime number. So $a + 1$ is prime, and a is one less than a prime number. We now list possible a values and count all $p^a < 2025$.

a	$a+1$ (prime)	Possible p values (since $p^a < 2025$)
2	3	$p < 45 \Rightarrow 14$ primes: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43$
4	5	$p < 6.7 \Rightarrow 2, 3, 5$ (3 primes)
6	7	$p < 3.4 \Rightarrow 2, 3$ (2 primes)
10	11	$p < 2.2 \Rightarrow 2$ (1 prime)

Adding them up, we get $14 + 3 + 2 + 1 = \boxed{20}$.

T8. Angela is standing on the bottom left square of a 3×3 grid. She wants to reach the top right square by moving only right or up one square at a time. Find the number of ways Bill can fill each square with the letter `a`, `b`, or `c`, such that every square contains exactly one letter and no matter what path Angela takes she will always encounter the letters `a`, `b`, `c` consecutively, in that order.

Proposed by: Oliver Zhang

Answer: $\boxed{77}$

Solution: The key idea is to consider the letters Angela encounters on the following two paths:

	X	X
X	X	
X		

		X
	X	X
X	X	

Note that Angela can only encounter the letters `abc` in sequence once no matter the path she takes. By simple logic, one can show that fixing the position of `abc` on one of the two paths above also fixes the position of `abc` on the other path. Specifically, there are 3 cases:

b	c	
a	b	

	c	
a	b	c
	o	

	b	c
	a	b

Now we consider placing the position of `orz` on the other two paths for each of the three cases. In the first case, there are four ways we can do so:

c		
b	c	
a	b	c

c		c
b	c	b
a	b	a

a	b	c
b	c	
a	b	c

a	b	c
b	c	b
a	b	o

The blank grids can be filled with any letter, so there are $3^3 + 3 + 3 + 1 = 34$ ways.

Similarly, there are four ways to place `orz` in the third case, and there are also 34 ways:

a	b	c
	a	b
		a

a	b	c
	a	b
a	b	c

c	b	c
b	a	b
a		a

c	b	c
b	a	b
a	b	z

In the second case, notice that the position of `orz` is fixed on the two other paths as well:

b	c	
a	b	c
	a	r

So there are $3^2 = 9$ ways.

To check that we there is no overlap in the casework above, observe that the letters along the top-left to bottom-right diagonal in every case are all distinct. Hence, the answer is $34 + 34 + 9 = \boxed{77}$.

T9. Let the *trimonic mean* of two positive integers a and b be $\frac{3}{\frac{1}{a}+\frac{1}{b}}$. Find the number of positive integers less than 100 that can be expressed as the trimonic mean of two distinct positive integers.

Proposed by: Oliver Zhang

Answer: $\boxed{84}$

Solution: Suppose $k = \frac{3}{\frac{1}{a}+\frac{1}{b}}$. Then

$$k = \frac{3ab}{a+b} \implies 3ab - ka - kb = 0 \implies (3a-k)(3b-k) = k^2.$$

Hence, if k^2 can be written as $k^2 = xy$, we have $a = \frac{k+x}{3}$ and $b = \frac{k+y}{3}$. This implies $k + x \equiv k + y \equiv 0 \pmod 3$, so $x \equiv y \pmod 3$. Also, $a \neq b \iff x \neq y$. We now do casework on $k \pmod 3$.

Case 1: $k \equiv 0 \pmod 3$.

Note that $k^2 \equiv 0 \pmod 9$, so if $k > 3$ then $x = 3$ and $y = \frac{k^2}{3}$ gives a valid pair $(a, b) = \left(\frac{k+3}{3}, \frac{k+k^2/3}{3}\right) = \left(1 + \frac{k}{3}, \frac{k}{3} + \frac{k^2}{9}\right)$.

Case 2: $k \equiv 2 \pmod 3$.

We have $k^2 \equiv 1 \pmod 3$, so $k + 1 \equiv k + k^2 \equiv 0 \pmod 3$. Thus, $x = 1$ and $y = k^2$ gives a valid pair $(a, b) = \left(\frac{k+1}{3}, \frac{k+k^2}{3}\right)$.

Case 3: $k \equiv 1 \pmod 3$.

For there to be a valid pair there must exist some $x \mid k$, where $x \equiv 2 \pmod 3$. Obviously if k is even we are done. If $k = 6r + 1$, however, there is no obvious conclusion. Therefore, we proceed by listing out those values of k:

$$k = 7, 13, 19, 25, 31, 37, 43, 49, 55, 61, 67, 73, 79, 85, 91, 97.$$

Note that all values of k except for 25, 49, 55, 85, and 91 are prime. It follows that all those prime k cannot be expressed as the trimonic mean of two distinct positive integers. For the remaining 5 values, note that 49 and 91 are also invalid since all their divisors are $1 \pmod 3$, but $5 \mid 25, 55, 85$, so those three values of k are valid. (Specifically, take $x = 5$ and $y = \frac{k^2}{5}$, so $(a, b) = \left(\frac{k+5}{3}, \frac{k+k^2/5}{3}\right)$.)

Combining the three cases, we see that the only invalid k are 1, 3, 49, 91, and the 11 other primes listed above. There are 99 integers between 1 and 99, inclusive, so the answer is $99 - 4 - 11 = \boxed{84}$.

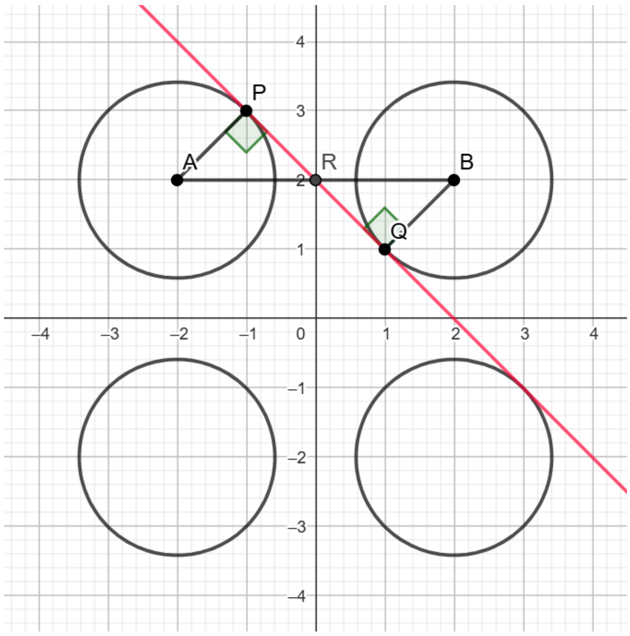
T10. Let $k > 1$ be the unique real number such that the graph of $(|x|-k)^2+(|y|-k)^2 = k$ in any 3 of the 4 quadrants share a common tangent line. The area of the region bound by the graph of $(|x|+k)^2+(|y|+k)^2 = k^4$ can be written in the form $\frac{a-b\sqrt{n}+c\pi}{d}$, where a, b, c, d, n are positive integers, n is square-free, and $\gcd(a, b, c, d) = 1$. Compute $a + bn + c + d$.

Proposed by: Oliver Zhang

Answer: $\boxed{211}$

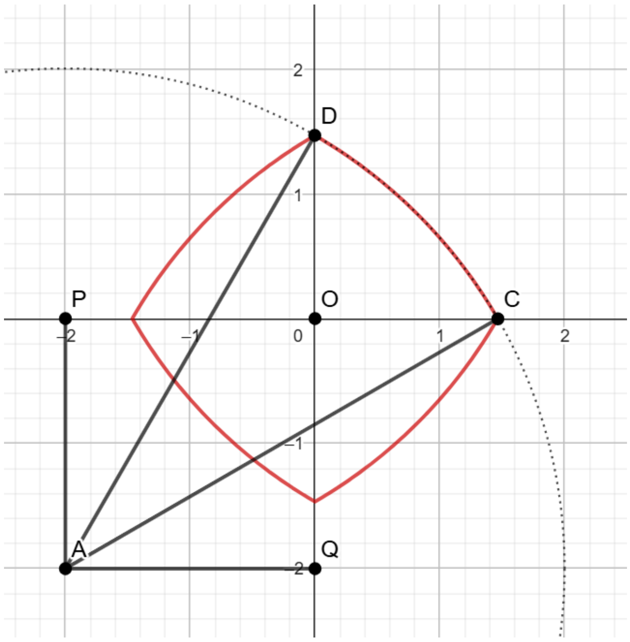
Solution: We will first find k. Note that the graph of $(|x| - k)^2 + (|y| - k)^2 = k$ consists of 4 circles with radii $\sqrt{k}$ centered at $(\pm k, \pm k)$. By symmetry, as long as some three circles share a common tangent line, any three circles will share a common tangent line.

Consider the three circles and their common tangent line shown below. By symmetry, the tangent line must be in the form $y = -x + t$.



Labeling the centers and the tangency points as shown above, note that $\triangle APR$ and $\triangle BQR$ are both isosceles right. $AB = 2k$, so we have $AR = \sqrt{2}AP \implies k = \sqrt{2k} \implies k = 2$.

Now we look at the area bound by $(|x| + k)^2 + (|y| + k)^2 = k^4$. Similarly, this graph consists of 4 circles with radii k^2 centered at $(\pm k, \pm k)$.



Consider the part of the graph in the first quadrant. This portion is an arc of the circle $(x+2)^2 + (y+2)^2 = 16$. Let the center of this circle be A and its x and y intercepts be C and D, respectively. Let P and Q be the projection of A onto the x and y-axis, respectively. Note that $AD = 2AQ$ and $\angle AQD = 90°$, so $\angle ADQ = 30°$ and thus $\angle PAD = 30°$. Similarly, $\angle DAQ = 30°$, so $\angle DAC = 30°$. Now, $QD = 2\sqrt{3}$, so $OD = 2\sqrt{3} - 2$. Then,

$$[O\widehat{CD}] = [A\widehat{CD}] - [AOD] - [AOC] = \frac{1}{12}\pi(4)^2 - 2(2\sqrt{3}-2) = \frac{4}{3}\pi + 4 - 4\sqrt{3}.$$

The answer is then

$$4[O\widehat{CD}] = \frac{16}{3}\pi + 16 - 16\sqrt{3} = \frac{48 - 48\sqrt{3} + 16\pi}{3} \implies \boxed{211}.$$

T11. A number is constructed using only the digits 1 and 2. We consider a number to be ”special” if the sum of its digits is 12. Two special numbers are considered equivalent if one can be obtained from the other by a cyclic shift of its digits. For example, 11221122 is equivalent to 12211221. How many non-equivalent special numbers are there?

Proposed by: Paige Zhu

Answer: $\boxed{31}$

Solution: A special number is a string of digits made of only 1 and 2 whose digit sum is 12. If the string contains k digits equal to 2, then the total number of digits is $n = 12 - k$. Two strings are considered equivalent if one is a cyclic shift of the other, so we must count binary necklaces of length n with exactly k twos.

Let $C_n(k)$ denote the number of distinct necklaces of length n with exactly k twos. We use Burnside's lemma. A rotation that shifts by an amount whose greatest common divisor with n equals g creates g cycles of positions, where each cycle has length $m = n/g$. This rotation fixes a string with exactly k twos only when m divides k. In this case the number of fixed strings equals the number of ways to choose which cycles carry twos.

This count equals

$$\binom{g}{k/m}.$$

There are $\varphi(n/g)$ such rotations, where φ is the Euler function. Hence

$$C_n(k) = \frac{1}{n} \sum_{\substack{g|n \\ m=n/g|k}} \varphi\left(\frac{n}{g}\right) \binom{g}{k/m}.$$

We compute $C_{12-k}(k)$ for all possible values of k from 0 to 6.

$$\begin{aligned}
k = 0,\ n = 12, &\qquad C_{12}(0) = 1. \\
k = 1,\ n = 11, &\qquad C_{11}(1) = 1. \\
k = 2,\ n = 10, &\qquad C_{10}(2) = 5. \\
k = 3,\ n = 9, &\qquad C_{9}(3) = 10. \\
k = 4,\ n = 8, &\qquad C_{8}(4) = 10. \\
k = 5,\ n = 7, &\qquad C_{7}(5) = 3. \\
k = 6,\ n = 6, &\qquad C_{6}(6) = 1.
\end{aligned}$$

The total number of distinct special numbers equals

$$1 + 1 + 5 + 10 + 10 + 3 + 1 = \boxed{31}.$$

T12. Let $P(x)$ be a monic polynomial of degree 6 with integer coefficients satisfying

$$\begin{aligned}
P(0) + P(2) &= 74, \\
P(-1) + P(3) &= 260, \\
P(-2) + P(4) &= 1770, \\
P(3) &= 230.
\end{aligned}$$

If $r_1, r_2, \ldots, r_6$ are the roots of $P(x)$, compute

$$(1 - r_1^2)(1 - r_2^2)(1 - r_3^2)(1 - r_4^2)(1 - r_5^2)(1 - r_6^2).$$

Proposed by: Jialai She

Answer: $\boxed{900}$

Solution: Observe that

$$(1 - r_1^2)(1 - r_2^2)(1 - r_3^2)(1 - r_4^2)(1 - r_5^2)(1 - r_6^2) = P(-1)P(1).$$

From the second and fourth equations, we must have $P(-1) = 30$. We now want to find $P(1)$.

Notice that the left sides of the given equations are of the form $P(1-t) + P(1+t)$. In particular, if we define

$$P(1 + t) = t^6 + a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0$$

then we have

$$P(1 - t) + P(1 + t) = 2(a_0 + a_2t^2 + a_4t^4 + a_6t^6).$$

Plugging in $t = 1, 2, 3$ into these equations then gives us:

$$\frac{P(0)+P(2)}{2} = a_0 + a_2 + a_4 + 1 = 37$$

$$\frac{P(-1)+P(3)}{2} = a_0 + 4a_2 + 16a_4 + 64 = 130$$

$$\frac{P(-2)+P(4)}{2} = a_0 + 9a_2 + 81a_4 + 729 = 885.$$

Solving these equations for a_0, a_2, and a_4 yields $a_0 = P(1) = 30$. Therefore, the answer is

$$P(-1)P(1) = 30 \cdot 30 = \boxed{900}.$$

T13. Let S be the set of integers n with $1 \le n \le 2000$ such that the greatest common divisor of $2^n + 1$ and $3^n + 1$ is divisible by 5 but not by 25, and such that $2^n + 1$ is divisible by 13. Compute the sum of all elements of S.

Proposed by: Jialai She

Answer: $\boxed{134664}$

Solution: Working in mod 5, we can see that the residues of 2^n and 3^n repeat with period 4. In particular, since $2^2 \equiv 3^2 \equiv -1 \pmod 5$, we must have that $n \equiv 2 \pmod 4$. Working modulo 25, one can check that $2^{10} \equiv 3^{10} \equiv -1 \pmod{25}$, and hence, we must have $n \not\equiv 10 \pmod{20}$.

Now, we consider $13|(2^n + 1)$. By Fermat's Little Theorem, we have $2^{12} \equiv 1 \pmod{13}$, and we can easily check that $2^6 = 64 \equiv -1 \pmod{13}$. Hence, we must have $n \equiv 6 \pmod{12}$.

Note that if $n \equiv 6 \pmod{12}$, then $n \equiv 2 \pmod 4$ must be true. So, the total solution set is just the integers n such that

$$n \equiv 6 \pmod{12}, \qquad n \not\equiv 10 \pmod{20}.$$

Working mod $\mathrm{lcm}(12, 20) = 60$, the solution set is simply

$$n \equiv 6, 18, 42, 54 \pmod{60}.$$

For each residue r, the terms ≤ 2000 form an arithmetic sequence with difference 60. The number of terms, k, and the sums of each arithmetic sequence are:

$$\begin{aligned} r = 6: \quad & k = 34, \quad \sum = 34 \cdot 996 = 33864, \\ r = 18: \quad & k = 34, \quad \sum = 34 \cdot 1008 = 34272, \\ r = 42: \quad & k = 33, \quad \sum = 33 \cdot 1002 = 33066, \\ r = 54: \quad & k = 33, \quad \sum = 33 \cdot 1014 = 33462. \end{aligned}$$

Adding these four sums gives

$$33864 + 34272 + 33066 + 33462 = \boxed{134664}.$$

T14. [**45**] Consider the infinite sequence a_n consisting of only the positive integers 3 and 4, such that $a_1 = 3$, and a_n is equal to the number of 4's between the n^{th} and $(n+1)^{\text{st}}$ 3 in the sequence. For example, the next four terms are

$$a_2 = 4, \quad a_3 = 4, \quad a_4 = 4, \quad a_5 = 3.$$

Find the number of 3's in the first 2640 terms of the sequence.

Proposed by: Brandon Xu

Answer: $\boxed{552}$

Solution: Let $s_1 = 3$. Subsequently, we define s_k to be the number with the leading digit 3, followed by the number generated when each digit 3 in s_{k-1} is replaced with the digits 4443 and each digit 4 is replaced with 44443. So, we have

$$s_2 = 34443,$$

$$s_3 = 344434444344443444434443$$

and so on. Note that the digits of s_n correespond to the leading terms of a_n. Let d_k be the number of digits in s_k, and b_k, c_k, to denote the number of 3's and 4's respectively in the digits of s_k. We can write the following recurrence relations:

$$\begin{aligned} b_n &= 1 + d_{n-1} \\ c_n &= 3b_{n-1} + 4c_{n-1} \end{aligned}$$

From these equations, we get that

$$\begin{aligned} d_n = b_n + c_n &= d_{n-1} + 1 + 3(b_{n-1} + c_{n-1}) + c_{n-1} \\ &= 4d_{n-1} + d_{n-1} - b_{n-1} + 1 \\ &= 5d_{n-1} - d_{n-2}. \end{aligned}$$

Note that $d_1 = 1, d_2 = 5, d_3 = 24$. From this, we can calculate

$$\begin{aligned} d_4 &= 115 \\ d_5 &= 551 \\ d_6 &= 2640. \end{aligned}$$

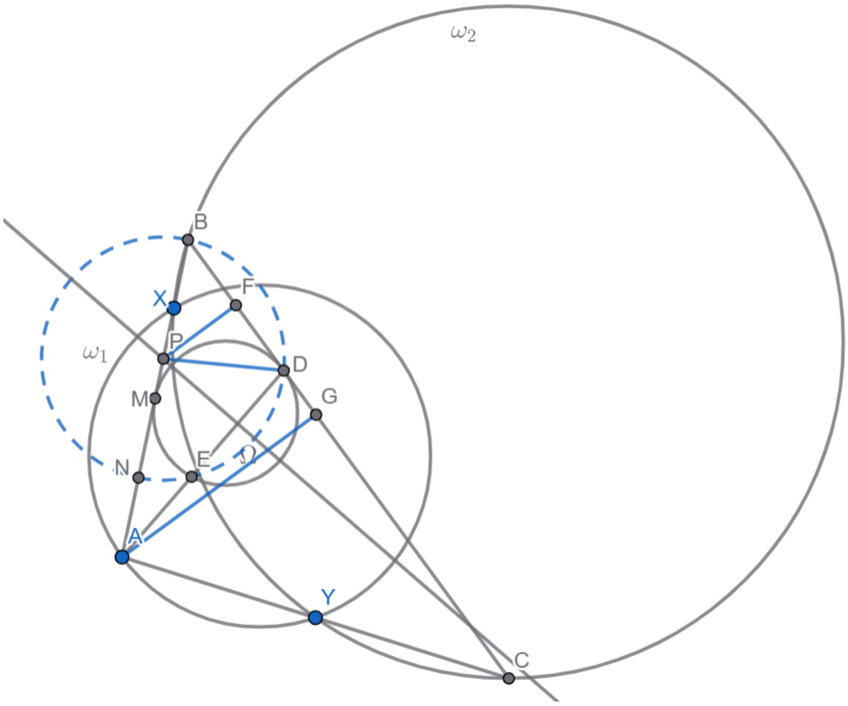
Hence, it turns out that the first 2640 terms of a_n correspond to the digits of s_6. Since $b_n = 1 + d_{n-1}$, we have $b_6 = 1 + 551 = \boxed{552}$.

T15. Let ω_1, ω_2 be two circles that intersect at X and Y, and let A be a point on ω_1. Lines AX and AY intersect ω_2 at B and C, respectively, such that $AY = YC = 20$ and $AX = 25$. M is the midpoint of AB and N is the midpoint of AM. Denote by Ω the circle passing through M tangent to segments AB and BC. D is the tangency point of Ω to $\overline{BC}$, and $\overline{AD}$ intersects Ω again at E. Suppose that the perpendicular bisector of DE passes through the midpoint of BN. Then AE^2 can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Oliver Zhang

Answer: $\boxed{775}$

Solution:



We claim that $BDEN$ is cyclic. To show this, note that by power of a point

$$AE \cdot AD = AM^2 = \frac{1}{2}AM \cdot 2AM = AN \cdot AB,$$

so $AE \cdot AD = AN \cdot AB$ which implies the claim.

Suppose P is the midpoint of BN. Then P is the center of $(BDEN)$. By Power of a Point, we have $AX \cdot AB = AY \cdot AC$, so $AB = \frac{20 \cdot 40}{25} = 32$. Then $AM = 16$ and $AN = 8$, so $BN = 24$ and $BP = 12$. Also, $BD = BM = 16$.

Notice that $\triangle BPD$ is isosceles. Let F and G be the feet of the altitudes to BC from P and A, respectively. F is the midpoint of BD, so $BF = 8$ and $PF = \sqrt{12^2 - 8^2} = 4\sqrt{5}$. $\triangle BPF \sim \triangle BAG$, so $BG = \frac{32}{12} \cdot 8 = \frac{64}{3}$ and $AG = \frac{32}{12} \cdot 4\sqrt{5} = \frac{32\sqrt{5}}{3}$. Then $DG = BG - BD = \frac{16}{3}$, and $AD = \sqrt{\left(\frac{32\sqrt{5}}{3}\right)^2 + \left(\frac{16}{3}\right)^2} = \sqrt{\frac{5376}{9}} = \frac{16\sqrt{21}}{3}$. Finally by PoP again, $AE \cdot AD = AM^2$, so $AE \cdot \frac{16\sqrt{21}}{3} = 16^2 \implies AE = \frac{48}{\sqrt{21}} \implies AE^2 = \frac{768}{7} \implies \boxed{775}$.