

MOAA 2026 Gunga Bowl Solutions

MATH OPEN AT ANDOVER

October 10th, 2026

- G1. The grid below is to be filled with real numbers such that the sum of the 3 numbers in any row, column, or diagonal are equal. Find the number in the center square.

	19	
	?	
17		13

Proposed by: Carolyn Cao

Answer: 11

Solution: Let the center square be x and the square below it be y :

	19	
	x	
17	y	13

The sum of the numbers in the middle column is equal to the sum of the numbers in the bottom row, so

$$19 + x + y = 17 + y + 13 \implies x = \boxed{11}.$$

- G2. Yoll is trying to escape from Eddie! Yoll starts at $(0, 0)$ and moves in the positive y -direction at a speed of 1 unit per second. Eddie starts at $(3, 0)$ and can move in any direction at a speed of 2 units per second. If Eddie chases Yoll optimally, he will catch up to Yoll after s seconds. Find s^2 .

Proposed by: Yoll Feng

Answer: 3

Solution: Let $E = (3, 0)$. Obviously, Eddie should move in a straight line towards some point on the positive y -axis in such a way that his path intersects with Yoll's on the y -axis. In s seconds, Yoll moves to $Y = (0, s)$, so in an ideal case we would like the distance from Y to E to be $2s$. If we let O denote the origin, this implies that $YE = 2OY$, so $\triangle OYE$ is a $30 - 60 - 90$ triangle, implying $s\sqrt{3} = 3 \implies s = \sqrt{3} \implies s^2 = \boxed{3}$.

- G3. Find the unique real number x satisfying

$$\sqrt{x + \sqrt{x + 11}} + \sqrt{x - \sqrt{x + 11}} = 4.$$

Proposed by: Carolyn Cao

Answer: $\boxed{5}$

Solution:

Remark – Since answers must be integers and we are given that the solution x is unique, trying small values of x almost immediately gives the answer.

Square both sides of the equation directly:

$$\begin{aligned}(x + \sqrt{x + 11}) + (x - \sqrt{x + 11}) + 2\sqrt{(x + \sqrt{x + 11})(x - \sqrt{x + 11})} &= 16 \\ \implies 2x + 2\sqrt{x^2 - (x + 11)} &= 16 \\ \implies x + \sqrt{x^2 - x - 11} &= 8.\end{aligned}$$

Isolate the remaining radical and square a second time:

$$\begin{aligned}\sqrt{x^2 - x - 11} &= 8 - x \\ x^2 - x - 11 &= 64 - 16x + x^2 \\ 15x &= 75 \\ \implies x &= \boxed{5}.\end{aligned}$$

Indeed, plugging $x = 5$ back into the original expression, we see that

$$\sqrt{5 + \sqrt{16}} + \sqrt{5 - \sqrt{16}} = \sqrt{9} + \sqrt{1} = 4,$$

so this answer works.

- G4. Given that the product of the real roots of

$$\prod_{k=1}^{2026} (x^2 + kx + 2027)$$

can be written in the form p^n , where p is a prime number, find n .

Proposed by: Carolyn Cao

Answer: $\boxed{1936}$

Solution: We observe that $x^2 + kx + 2027$ has two distinct real roots if and only if $k^2 - 4 \cdot 2027 > 0$, or $k > 2 \cdot \sqrt{2027} > 2 \cdot 45 = 90$. It is easy to see that $91^2 > 4 \cdot 2027$, so we get $91 \leq k \leq 2026$, a total of 1936 values. Since any two such pairs of solutions are distinct and have product 2027 by Vieta, the entire product is equal to 2027^{1936} . 2027 is prime, so the answer is $\boxed{1936}$.

- G5. The 2026 MOAA Board consists of 6 Seniors and 6 Uppers. If Walleye wants to form a committee such that there are no more Uppers than Seniors, find the number of ways Walleye can form a non-empty committee.

Proposed by: Paige Zhu

Answer: $\boxed{2509}$

Solution: There are $2^{12} = 4096$ committees Walleye can form without any restrictions, and

$$\binom{6}{0}^2 + \binom{6}{1}^2 + \cdots + \binom{6}{6}^2 = 1 + 36 + 225 + 400 + 225 + 36 + 1 = 924$$

of those have exactly the same number of Uppers as Seniors. By symmetry, the number of committees with strictly fewer Uppers than Seniors is equal to the number of committees with strictly fewer Seniors than Uppers. Therefore, there are

$$\frac{1}{2} \cdot (4096 - 924) = 1586$$

possibly nonempty committees Walleye can form which have strictly fewer Uppers than Seniors. We add back 924 for the cases where there are exactly as many Uppers as Seniors, and subtract 1 for the added empty committee to get the answer

$$1586 + 924 - 1 = \boxed{2509}.$$

- G6. Three mutually externally tangent circles are drawn such that they all share a single common external tangent line. The two larger circles have radii of 4 and 9. If the radius of the smallest circle can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

Proposed by: Paige Zhu

Answer: $\boxed{61}$

Solution: Let Ω_1 and Ω_2 be the circles with radii 4 and 9, respectively, and let γ be the small circle. Let the radius of the small circle be r . Let O_1 , O_2 , and O_3 be the centers of Ω_1 , Ω_2 , and γ , respectively. Finally, let X , Y , and Z be the points of tangency of the line with Ω_1 , Ω_2 , and γ , respectively.

Let R be the foot of the altitude from O_1 to O_2Y . Since $O_1O_2 = 4 + 9 + 13$ and $O_2R = 9 - 4 = 5$, we have by the Pythagorean theorem that $AR = 12$. Since $AXYR$ is a rectangle, we have $XY = 12$ as well.

Now, let P and Q be the feet from O_3 to O_1X and O_2Y , respectively. Pythagorean theorem on $\triangle O_1O_3P$ and $\triangle O_2O_3P$ gives

$$O_3P = \sqrt{(4+r)^2 - (4-r)^2} = 4\sqrt{r} \quad \text{and} \quad O_3Q = \sqrt{(9+r)^2 - (9-r)^2} = 6\sqrt{r}.$$

$XYQP$ is again a rectangle, so $O_3P + O_3Q = XY$ and hence

$$10\sqrt{r} = 12 \implies \sqrt{r} = \frac{6}{5} \implies r = \frac{36}{25} \implies 36 + 25 = \boxed{61}.$$

- G7. Find the smallest positive integer that can be expressed as the product of two not necessarily distinct perfect squares in exactly 7 different ways.

(Clarification in test: the expressions $a^2 \cdot b^2$ and $b^2 \cdot a^2$ are considered the same.)

Proposed by: Oliver Zhang

Answer: $\boxed{36864}$

Solution: Obviously any such number n with the given property is a perfect square. Let the prime factorization of n be $n = (p_1^2)^{a_1} (p_2^2)^{a_2} \cdots (p_k^2)^{a_k}$. Then n has a total of $T = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$ perfect square factors.

If T is even, then every perfect square factor k can be paired with a different perfect square factor $\frac{n}{k}$ that multiply to n , so there are $\frac{T}{2}$ ways to write n as the product of two perfect squares.

If T is odd, then every a_i is even, so n must be a perfect 4th power (and $\sqrt{n}$ is a perfect square that pairs with itself), thus there are $\frac{T+1}{2}$ ways.

Hence, either $\frac{T}{2} = 7$ or $\frac{T+1}{2} = 7$, from where we get $T = 14$ or $T = 13$. In the previous case,

$$\begin{cases} 14 = (13 + 1) \implies n = (p^2)^{13} = p^{26} \\ 14 = (6 + 1)(1 + 1) \implies n = (p^2)^6(q^2)^1 = p^{12}q^2 \end{cases}$$

and in the latter case,

$$13 = (12 + 1) \implies n = (p^2)^{12} = p^{24}.$$

Therefore, $n = p^{26}$, p^{24} , or $p^{12}q^2$. Obviously $n = p^{12}q^2$ yields the least possible n , so $n = 2^{12} \cdot 3^2 = 4096 \cdot 9 = \boxed{36864}$.

- G8. Let ABC be a triangle. The angle bisector of $\angle BAC$, the perpendicular from B to side AC , and the perpendicular bisector of AB intersect at a single point X . If $AB = 26$, find the largest possible length of BC for which the length of AC is an integer.

Proposed by: Oliver Zhang

Answer: $\boxed{254}$

Solution: Let D be the foot from B to AC , and M be the midpoint of AB .

Claim. $\angle BAC = 60^\circ$.

Proof. Obviously $\triangle AXM \cong \triangle BXM$ and $\triangle AXM \cong \triangle AXD$, so $\angle AXD = \angle AXM = \angle BXM$. Those three angles sum to $\angle DXB = 180^\circ$, so they must each be 60° . Then $\angle DAX = \angle MAX = 30^\circ$, implying the result. $\square$

Now, $AD = 13$, so $BD = 13\sqrt{3}$. Let $DC = x$ and $BC = k$. Then AC is an integer if and only if x is an integer. By the Pythagorean Theorem,

$$x^2 + (13\sqrt{3})^2 = k^2 \implies (k+x)(k-x) = 3 \cdot 13^2.$$

The maximum value of k occurs when $k+x$ is maximized, so we let $k+x = 3 \cdot 13^2 = 507$ and $k-x = 1$. Then $k = \frac{1}{2}(507+1) = \boxed{254}$.

- G9. Let $P(x)$ be a monic polynomial of degree 6. Given that $P(1) = 1$, $P(2) = 2$, $P(3) = 5$, $P(4) = 12$, $P(5) = 25$, and $P(6) = 46$, find the value of $P(7)$.

Proposed by: Paige Zhu

Answer: $\boxed{77}$

Solution: We use finite differences:

$$\begin{array}{cccccc} 1 & 2 & 5 & 12 & 25 & 46 \implies \boxed{77} \\ 1 & 3 & 7 & 13 & 21 & \implies 31 \\ 2 & 4 & 6 & 8 & \implies 10 \\ 2 & 2 & 2 & \implies 2 \end{array}$$

- G10. Let N be the answer to this problem, and let M be the largest proper divisor of N . Given that M is a multiple of 5, find the largest proper divisor of M^2 .

(Clarification in test: N is a positive integer.)

Proposed by: Oliver Zhang

Answer: 125

Solution: We rewrite the problem as follows: Find positive integers N , M such that M is the largest proper divisor of N , N is the largest proper divisor of M^2 , and M is a multiple of 5.

Claim. $N = p^3$ for some prime p .

Proof. Let $N = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ be the prime factorization of N , with $p_1 < p_2 < \dots < p_k$. Then $M = p_1^{a_1-1} p_2^{a_2} \dots p_k^{a_k}$, and $M^2 = p_1^{2a_1-2} p_2^{2a_2} \dots p_k^{2a_k}$, so the largest proper divisor of M^2 is $p_1^{2a_1-3} p_2^{2a_2} \dots p_k^{2a_k}$. Therefore, we need $p_1^{2a_1-3} p_2^{2a_2} \dots p_k^{2a_k} = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, which implies $2a_1 - 3 = a_1 \implies a_1 = 3$ and $2a_i = a_i \implies a_i = 0$ for all $i > 1$. Therefore, $N = p_1^3 p_2^0 \dots p_k^0 = p_1^3$. $\square$

By the claim above, we see that $M = p^2$, so since M is a multiple of 5 we must have $p = 5$ and $N = 5^3 = \boxed{125}$.

- G11. A 20×26 grid is filled with nonnegative integers. Every entry in the first row and first column is 0. Whenever one moves one square to the right or one square down, the new entry is either equal to the old entry or exactly 1 larger. Find the greatest possible sum of all 520 entries in the grid.

Proposed by: Yoll Feng

Answer: 3610

Solution: Consider an entry in row i and column j . Moving down from the first row shows that it is at most $i - 1$, and moving right from the first column shows that it is at most $j - 1$. Hence, $a_{ij} \leq \min(i - 1, j - 1)$. This bound is attained everywhere simultaneously by setting $a_{ij} = \min(i - 1, j - 1)$.

Therefore, the desired maximum sum is

$$\sum_{r=0}^{19} \sum_{c=0}^{25} \min(r, c).$$

For a fixed r with $0 \leq r \leq 19$,

$$\sum_{c=0}^{19} \min(r, c) = \frac{r(r-1)}{2} + (26-r)r = \frac{51r - r^2}{2}.$$

Hence, summing over all such r , the maximum simplifies to

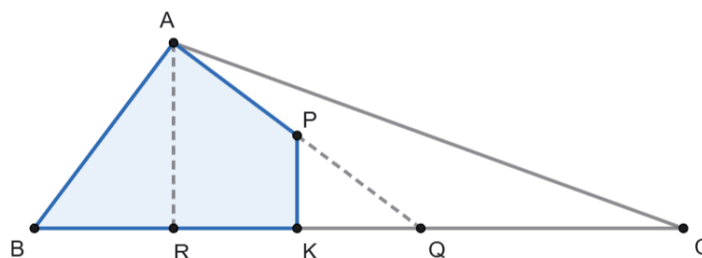
$$\frac{1}{2} \left(51 \sum_{r=0}^{19} r - \sum_{r=0}^{19} r^2 \right) = \frac{1}{2} (51 \cdot 190 - 2470) = \boxed{3610}.$$

- G12. Let ABC be a triangle with $AB = 15$, $BC = 42$, and area 252. Point K lies on segment $\overline{BC}$, and point P lies within triangle ABC such that $PA \perp AB$ and $PK \perp BC$. Suppose that the area of quadrilateral $ABKP$ is exactly $\frac{1}{2}$ the area of triangle ABC . Find the area of triangle APC .

Proposed by: Oliver Zhang

Answer: 51

Solution:



Extend AP to intersect BC at point Q , and let R be the foot from A to BC . We find $AR = 12$, so $BR = 9$, and by $\triangle ABR \sim \triangle QAR$ we get $QR = 16$, $AQ = 20$.

Now, since $PL \parallel AR$ we have $\triangle PKQ \sim \triangle ARQ$. Hence, let $KQ = 4k$, so $PK = 3k$ and $PQ = 5k$, so $[PKQ] = 6k^2$. $[ABQ] = \frac{1}{2} \cdot (9 + 16) \cdot 12 = 150$, so we need $150 - 6k^2 = \frac{1}{2} \cdot 252 = 126$, which implies $k = 2$. Hence, $KQ = 8 = \frac{1}{2}RQ$, so K and P are the midpoints of RQ and AQ , respectively, and therefore $[APC] = \frac{1}{2}[AQC]$.

Finally, $CQ = 42 - 25 = 17$, so $[AQC] = \frac{1}{2} \cdot 17 \cdot 12 = 102$. Then $[APC] = \frac{1}{2} \cdot 102 = \boxed{51}$.

- G13. Gunga has some red, green, and blue balls, with one of each color in a bag and one of each color outside the bag. He repeatedly draws one ball randomly from the bag. If it is red, green, or blue, he puts back a green, blue, or red ball, respectively, and stops once he doesn't have any correctly colored balls left outside the bag. Find the expected number of balls Gunga will draw from the bag.

Proposed by: Oliver Zhang

Answer: 9

Solution: Consider all the possible arrangements of balls in the bag. Since the colors are cyclically symmetric, we can treat certain states as the same, for example RRG and GGB . With some casework, we can quickly see that there are only three distinct cases: RRG , RRB , and RGB . Let $\mathbb{E}(XYZ)$ denote the expected number of balls Gunga will draw until the process terminates, given that the bag currently is in the state XYZ . We can then do the following short casework:

- From RRG , Gunga has a $\frac{2}{3}$ chance of replacing R with G to get to $GGR = RRB$, or a $\frac{1}{3}$ chance of replacing G with B to also get to RRB .
- From RRB , Gunga has a $\frac{2}{3}$ chance of replacing R with G to get to RGB , or a $\frac{1}{3}$ chance of drawing the blue ball and terminating the process.
- From RGB , no matter what ball Gunga draws, he will always end up in a state equivalent to RRG .

Therefore,

$$\begin{cases} \mathbb{E}(RGB) = \mathbb{E}(RRG) + 1 & (1) \\ \mathbb{E}(RRG) = \mathbb{E}(RRB) + 1 & (2) \\ \mathbb{E}(RRB) = \frac{2}{3}\mathbb{E}(RGB) + \frac{1}{3} \cdot 0 + 1 & (3) \end{cases}$$

Substituting (3) into (2) and then into (1), we get

$$\mathbb{E}(RGB) = \frac{2}{3}\mathbb{E}(RGB) + 3 \implies \mathbb{E}(RGB) = \boxed{9}.$$

G14. Let

$$P = \frac{2^3 - 1}{2^3 + 1} \cdot \frac{3^3 - 1}{3^3 + 1} \cdot \frac{4^3 - 1}{4^3 + 1} \cdot \dots \cdot \frac{2026^3 - 1}{2026^3 + 1}.$$

Let Q be the smallest positive integer for which PQ is an integer. Find the sum of all distinct prime divisors of Q .

Proposed by: Ishan Gandhi

Answer: $\boxed{3040}$

Solution: For any integer k ,

$$k^3 - 1 = (k - 1)(k^2 + k + 1) \quad \text{and} \quad k^3 + 1 = (k + 1)(k^2 - k + 1),$$

thus

$$\frac{k^3 - 1}{k^3 + 1} = \frac{(k - 1)(k^2 + k + 1)}{(k + 1)(k^2 - k + 1)}.$$

Note that the $k + 1$ term in the denominator cancels with the $k - 1$ term in the numerator 2 terms after, so the product telescopes to

$$\begin{aligned} P &= \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot 2025}{3 \cdot 4 \cdot 5 \cdot \dots \cdot 2027} \cdot \frac{2^2 + 2 + 1}{2^2 - 2 + 1} \cdot \frac{3^2 + 3 + 1}{3^2 - 3 + 1} \cdot \dots \cdot \frac{2026^2 + 2026 + 1}{2026^2 - 2026 + 1} \\ &= \frac{1 \cdot 2}{2026 \cdot 2027} \cdot \frac{2^2 + 2 + 1}{2^2 - 2 + 1} \cdot \frac{3^2 + 3 + 1}{3^2 - 3 + 1} \cdot \dots \cdot \frac{2026^2 + 2026 + 1}{2026^2 - 2026 + 1}. \end{aligned}$$

Furthermore, notice that

$$k^2 - k + 1 = (k - 1)^2 + (k - 1) + 1,$$

so the $(k^2 - k + 1)$ term in the denominator in one term cancels with the $(k^2 + k + 1)$ term in the numerator of the previous term. Therefore, the product telescopes further to

$$P = \frac{1 \cdot 2}{2026 \cdot 2027} \cdot \frac{2026^2 + 2026 + 1}{2^2 - 2 + 1} = \frac{2 \cdot (2026^2 + 2026 + 1)}{3 \cdot 2026 \cdot 2027}.$$

Since $2026 \equiv 1 \pmod{3}$, we have $2026^2 + 2026 + 1 \equiv 1 + 1 + 1 \equiv 0 \pmod{3}$. Also, $2026 \equiv -1 \pmod{2027}$, so $2026^2 + 2026 + 1 \equiv 1 - 1 + 1 \equiv 1 \pmod{2027}$. Hence, the numerator is divisible by 3 but not 2027, and it is obviously not divisible by 1013. Hence, $Q = 1013 \cdot 2027$, both of which are prime, and the answer is $1013 + 2027 = \boxed{3040}$.

G15. Ellie is given a cubic polynomial $P(x) = x^3 - 9x^2 + 15x - C$ where C is a integer. She is told that this polynomial has three distinct positive real roots. Find the maximum possible value of C .

Proposed by: Paige Zhu

Answer: $\boxed{6}$

Solution: Let us first consider the case where $P(x)$ has exactly two positive real roots. Then $P(x)$ factors into $(x - a)^2(x - b)$, so we have

$$x^3 - (2a + b)x^2 + (a^2 + 2ab)x - a^2b = x^3 - 9x^2 + 15x - C$$

from where we get

$$\begin{cases} 2a + b = 9 \\ a^2 + 2ab = 15 \end{cases} \implies b = 9 - 2a \implies a^2 + 2a(9 - 2a) = 15 \implies a = 1 \text{ or } 5.$$

Hence, either $P(x) = (x - 1)^2(x - 7)$ or $P(x) = (x - 5)^2(x + 1)$, corresponding to $C = 7$ and $C = -25$, respectively. Noting the graph of the first equation, we see quickly that $C < 7$. Furthermore, since $(0 - 1)^2(0 - 7) < -1$, we have that $(x - 1)^2(x - 7) + 1$ has three distinct positive roots, implying that $C = 6$ is valid. Hence, the maximum possible value of C is $\boxed{6}$.

G16. Suppose that real numbers x, y, z satisfy

$$\frac{1}{2} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) = \frac{1}{xyz} = x + y + z - xyz.$$

The smallest possible value of $x^2 + y^2 + z^2$ can be written as $\frac{m}{n}$, where m and n are positive co-prime integers. Find $m + n$.

Proposed by: Oliver Zhang

Answer: $\boxed{13}$

Solution: Let x, y, z be the roots of a cubic $x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3$, where, by Vieta,

$$\sigma_1 = x + y + z, \quad \sigma_2 = xy + yz + zx, \quad \sigma_3 = xyz.$$

The expression can be rewritten as

$$\frac{1}{2} \frac{\sigma_2}{\sigma_3} = \sigma_1 - \sigma_3, \quad \frac{1}{\sigma_3} = \sigma_1 - \sigma_3.$$

The second equality motivates setting $k = \sigma_3$, so that $\sigma_1 = k + \frac{1}{k}$. Furthermore we have

$$\frac{1}{2} \frac{\sigma_2}{\sigma_3} = \sigma_1 - \sigma_3 \implies \frac{1}{2} \frac{\sigma_2}{k} = \frac{1}{k} \implies \sigma_2 = 2.$$

Therefore, we have that our cubic is equal to

$$x^3 - \left(k + \frac{1}{k}\right)x^2 + 2x - k.$$

We want to minimize

$$x^2 + y^2 + z^2 = \sigma_1^2 - 2\sigma_2 = k^2 + \frac{1}{k^2} - 2 = \left(k - \frac{1}{k}\right)^2 \quad (1)$$

However, we still need that the cubic we got has 3 real roots, so there's still some work to do. Firstly, note that $x = k$ is a root:

$$k^3 - \left(k + \frac{1}{k}\right)k^2 + 2k - k = k^3 - k^3 + k + 2k - k = 0,$$

so after multiplying everything by k we can factor the cubic into

$$(x - k)(kx^2 - x + k).$$

The quadratic part has solutions if and only if $1 - 4k^2 \geq 0 \iff |k| \leq \frac{1}{2}$.

Finally, notice that (1) strictly increases when $|k|$ decreases if $|k| \leq 1$ (since $k = \frac{1}{k}$ if and only if $|k| = 1$), so the expression attains its minimum at $k = \frac{1}{2}$, where it evaluates to $(\frac{1}{2} - 2)^2 = (\frac{3}{2})^2 = \frac{9}{4} \implies \boxed{13}$.

- G17. Let f be a polynomial with nonnegative integer coefficients. Suppose that $f(1) = 11$ and $f(f(1) + 1) = 69696$. Find $f(f(1) - 1)$.

Proposed by: Oliver Zhang

Answer: $\boxed{34400}$

Solution: Since none of the coefficients of f can be greater than 11, the value of $f(f(1) + 1) = f(12)$ can be seen as a base-12 number with the coefficients of f as its digits.* Therefore, since $(69696)_{10} = (34400)_{12}$, $f(x) = 3x^4 + 4x^3 + 4x^2$ and $f(10) = \boxed{34400}$.

* This can be seen more clearly by the expression $f(x) = \sum_{i=0} c_i x^i$. If $x = b$ and $0 \leq c_i < b$ for all i , then this expression is equivalent to the decimal expansion of the base b number $\overline{\dots c_2 c_1 c_0}$.

- G18. Compute

$$\sum_{x=1}^9 \sum_{y=1}^{x^2} \sum_{z=\lceil \sqrt{y} \rceil}^x \frac{y}{z^2 x},$$

where $\lceil n \rceil$ denotes the smallest integer greater than or equal to n .

Proposed by: Oliver Zhang

Answer: $\boxed{64}$

Solution: The idea is to swap the order of summation. Observe that

$$z \geq \lceil \sqrt{y} \rceil \iff y \leq z^2$$

for positive integers y and z . Therefore, by drawing a simple graph we quickly see that

$$\sum_{y=1}^{x^2} \sum_{z=\lceil \sqrt{y} \rceil}^x \frac{y}{z^2 x} = \sum_{z=1}^x \sum_{y=1}^{z^2} \frac{y}{z^2 x}.$$

The rest is simple:

$$\begin{aligned} \sum_{x=1}^9 \sum_{y=1}^{x^2} \sum_{z=\lceil \sqrt{y} \rceil}^x \frac{y}{z^2 x} &= \sum_{x=1}^9 \sum_{z=1}^x \sum_{y=1}^{z^2} \frac{y}{z^2 x} \\ &= \sum_{x=1}^9 \sum_{z=1}^x \frac{1}{z^2 x} \cdot \frac{z^2(z^2 + 1)}{2} \\ &= \frac{1}{2} \sum_{x=1}^9 \frac{1}{x} \sum_{z=1}^x (z^2 + 1) \end{aligned}$$

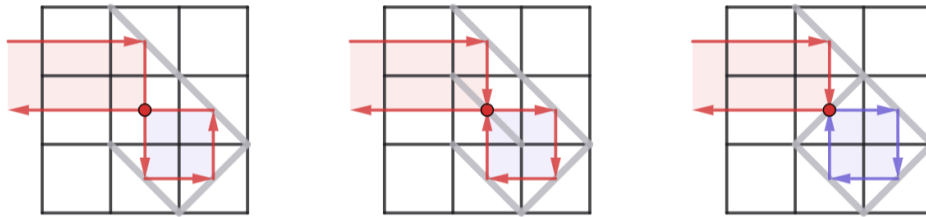
$$\begin{aligned}
 &= \frac{1}{2} \sum_{x=1}^9 \frac{1}{x} \cdot \left(\frac{x(x+1)(2x+1)}{6} + x \right) \\
 &= \frac{1}{12} \sum_{x=1}^9 (2x^2 + 3x + 7) \\
 &= \frac{1}{12} \left(2 \cdot \frac{9 \cdot 10 \cdot 19}{6} + 3 \cdot \frac{9 \cdot 10}{2} + 63 \right) \\
 &= \frac{1}{12} \cdot 768 = \boxed{64}.
 \end{aligned}$$

- G19. On a 3×7 board, a laser beam shoots in horizontally to the right from the top-left grid. Ishan can choose to place a double-sided mirror in some grids along exactly one of the two diagonals. A configuration of mirrors is called *return-to-sender* if the laser beam reaches the top-left grid a second time. Find the number of return-to-sender configurations.

Proposed by: Oliver Zhang

Answer: $\boxed{6887}$

Solution: Since each return-to-sender path is closed, it must enclose a region that is connected, with diagonal point connections allowed like the one in the example. Let us call those regions *semi-connected*. Hence, we may instead count the number of semi-connected regions in a 2×6 grid, where lattice points are the centers of the 3×7 board in the problem. There is one caveat, however:



As shown above, if the laser beam ever passes the same tile twice, there are two ways it could do so that yield the same semi-connected region, depending on whether there is a mirror on that tile or not. Furthermore, there can be only be one mirror placement on the tile, since one of the two orientations would cut off the two semi-connected regions, as shown on the right, as they cannot be connected anywhere else. Hence, if a semi-connected region has n diagonally connected components, it contributes 2^n to the final sum. We now proceed with recursion.

Let $f(n, k)$ denote the total number of return-to-sender paths corresponding to semi-connected regions in a $2 \times n$ grid that include the top-left square, has no empty column, and with k indicating the restriction on the rightmost column, as shown below:

$$\begin{cases} \text{region only includes the top square in the rightmost column} & k = 1 \\ \text{only the bottom square} & k = 2 \\ \text{both squares} & k = 3 \end{cases}$$

Hence, we get the three recursions:

$$\begin{cases} f(n, 1) = f(n-1, 1) + 2f(n-1, 2) + f(n-1, 3) \\ f(n, 2) = 2f(n-1, 1) + f(n-1, 2) + f(n-1, 3) \\ f(n, 3) = f(n-1, 1) + f(n-1, 2) + f(n-1, 3) \end{cases}$$

We seek the quantity

$$\sum_{i=1}^7 (f(i, 1) + f(i, 2) + f(i, 3)) = \sum_{i=1}^7 (f(i+1, 3)) = f(2, 3) + \cdots + f(8, 3).$$

Since we only care about the values of $f(n, 3)$, we can attempt to isolate a recursion for $f(n, 3)$ only:

$$\begin{aligned} f(n, 3) &= f(n-1, 1) + f(n-1, 2) + f(n-1, 3) \\ &= 3f(n-2, 1) + 3f(n-2, 2) + 2f(n-2, 3) + f(n-1, 3) \\ &= 4f(n-1, 3) - f(n-2, 3), \end{aligned}$$

where the second equality uses $f(n-1, 1) = f(n-2, 1) + 2f(n-2, 2) + f(n-2, 3)$ and similarly to $f(n-1, 2)$. This holds for all $n > 2$. We can easily compute $f(1, 3) = 1$ and $f(2, 3) = 2$, so we can calculate the first few values of $f(n, 3)$:

n	1	2	3	4	5	6	7	8
$f(n, 3)$	1	2	7	26	97	362	1351	5042

$$\text{so } 2 + 7 + 26 + 97 + 362 + 1351 + 5042 = \boxed{6887}.$$

- G20. Let ABC be a triangle with $AB = 77$, $BC = 66$, and $CA = 77$. Let M be the midpoint of AB , and O be the circumcenter of ABC . Suppose that point D lies on segment AC and point O' lies on ray MO such that $\frac{AC}{CD} = \frac{MO'}{MO} = 3$. Line DO' intersects the angle bisector of $\angle BAC$ at point G . Find the distance from A to line BG .

Proposed by: Carolyn Cao

Answer: $\boxed{13}$

Solution: Let E be the reflection of B across C . Then, D is the centroid of ABE and O' is the circumcenter of ABE . Therefore, X is the orthocenter of ABE . Let Y be the foot of the altitude from B to AE . Then, we must find AY . $AE = 132$, and we can compute the altitude from A to be equal to $22\sqrt{10}$. Then, the altitude from B is $24\sqrt{10}$, and the distance from A to the foot of the altitude is simply $\sqrt{77^2 - 24^2 \cdot 10} = 13$.

- G21. Find the number of ordered triples (p, q, r) , where p, q, r are primes no greater than 13, such that there exist two positive rational numbers u and v satisfying

$$u + v = \frac{q}{p} \quad \text{and} \quad uv = \frac{r}{p}.$$

Proposed by: Yoll Feng

Answer: $\boxed{11}$

Solution: Firstly, by Vieta, the numbers u and v are the roots of the quadratic $pz^2 - qz + r = 0$. Such rational numbers exist exactly when there exists some nonnegative integer s for which $q^2 - 4pr = s^2$. Since $q = 2$ would make the left side negative, q must be odd, and hence s has the same parity as q . Rearrange and factor:

$$(q - s)(q + s) = 4pr.$$

Hence the positive integers

$$A = \frac{q-s}{2} \quad \text{and} \quad B = \frac{q+s}{2}$$

satisfy $AB = pr$ and $A + B = q$.

Because p and r are prime, the unordered factor pair $\{A, B\}$ is either $\{p, r\}$ or $\{1, pr\}$.

- If $\{A, B\} = \{p, r\}$, then $q = p + r$. There are 6 such ordered triples with all primes at most 13:

$$(2, 5, 3), (3, 5, 2), (2, 7, 5), (5, 7, 2), (2, 13, 11), (11, 13, 2).$$

- If $\{A, B\} = \{1, pr\}$, then $q = pr + 1$. There are 5 such possibilities:

$$(2, 5, 2), (2, 7, 3), (3, 7, 2), (2, 11, 5), (5, 11, 2).$$

Thus the total is $6 + 5 = \boxed{11}$.

- G22. Alice is stacking a group of A RSM frisbees and B MehtA+ frisbees, where neither sponsor has more than twice as many frisbees as the other. If there are exactly 246 ways for her to stack the frisbees such that no three consecutive frisbees can be from the same sponsor, find $A + B$.

Proposed by: Paige Zhu

Answer: $\boxed{15}$

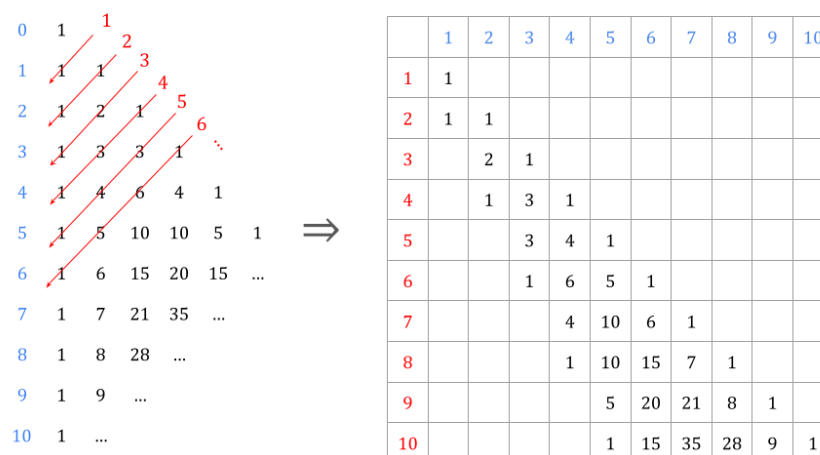
Solution: Any such stack can be separated into chunks of 1 or 2 frisbees, alternating in type, where the total number of chunks of RSM and MehtA+ frisbees differ by at most 1. Conversely, given a splitting of RSM and MehtA+ frisbees into n and m chunks, we have

$$\begin{cases} 1 \text{ way} & |n - m| = 1 \\ 2 \text{ ways} & n = m \\ 0 \text{ ways} & \text{otherwise.} \end{cases}$$

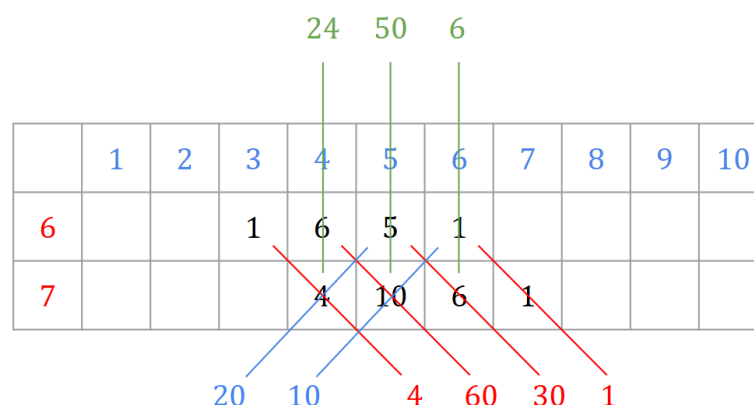
to combine them into one valid stack. Given X frisbees to be divided into n chunks of 1 or 2, we can first assign n of them to one per chunk, then there are $\binom{n}{X-n}$ ways to choose which chunks to place the $X - n$ remaining frisbees. The desired sum is then

$$\sum_n \binom{n}{A-n} \binom{n+1}{B-(n+1)} + \binom{n+1}{A-(n+1)} \binom{n}{B-n} + 2 \binom{n}{A-n} \binom{n}{B-n}.$$

One way of proceeding is to draw out Pascal's Triangle in its right triangle form:



As shown above, each family of desired coefficients $\binom{n}{x-n}$ lies along the top-right to bottom-left diagonals of Pascal's Triangle. Here, the red numbers indicate the value of N and the blue ones indicate x . Then, to find the total number of stackings given A and B , we use the rows corresponding to A and B in the table to the right by summing all products of “cross entries,” then doubling the sum of all products of “direct entries,” as shown below:



Hence the sum is

$$(20 + 10) + (4 + 60 + 30 + 1) + 2(24 + 50 + 6) = 285.$$

By fixing A and varying B , we quickly see that there are not many pairs (A, B) for which the sum is in the low-100s range. In particular, $A = 6$ and $B = 9$ yields

$$(5) + (30 + 100 + 21) + 2(25 + 20) = 246,$$

so $A + B = \boxed{15}$.

- G23. Isosceles triangle ABC with $AB = BC$ has its incircle γ centered at I with radius 3 touching side BC at D . Let M be the midpoint of BC , and suppose that lines MI and AD intersect on γ . If S is the area of ABC , compute S^2 .

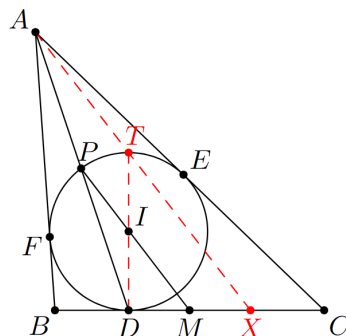
Proposed by: Oliver Zhang

Answer: $\boxed{3087}$

Solution: Let P be the intersection of MI and AD .

Claim. P is the midpoint of AD .

Proof. Let X be the tangency point of the A -excircle on BC , and let T be the point on γ diametrically opposite of D . There exists a dilation sending γ to the A -excircle centered at A , so it sends T to D , and hence A , T , and X are collinear. Now, it is well-known that $BD = XC$, so $DM = MX$. Furthermore, $DI = IT$, so $TX \parallel IM$ and hence $AX \parallel PM$. Using $DM = MX$ again, we see that PM is a mid-segment in $\triangle ADX$, implying the result. $\square$



Now, let γ be tangent to AB and AC at F and E , respectively. By Power of a Point, $AP \cdot AD = AF^2$, implying $AD : AF = 2 : \sqrt{2}$. Let $AD = 2x$ and $AF = AE = \sqrt{2}x$. From the isosceles condition we get $CE = CD = \sqrt{2}x$ as well. Let $BD = BF = d$. Then, by Stewart's Theorem on $\triangle ABC$, we have

$$\begin{aligned} d \cdot \sqrt{2}x \cdot (d + \sqrt{2}x) + 4x^2 \cdot (d + \sqrt{2}x) &= (d + \sqrt{2}x)^2 \cdot \sqrt{2}x + 8x^2 \cdot d \\ \sqrt{2}d^2 + 2xd + 4xd + 4\sqrt{2}x^2 &= \sqrt{2}(2x^2 + d^2 + 2\sqrt{2}xd) + 8xd \\ 2xd + 4\sqrt{2}x^2 &= 2\sqrt{2}x^2 + 8xd \\ 2\sqrt{2}x^2 &= 6xd \implies d = \frac{\sqrt{2}}{3}x. \end{aligned}$$

Plugging stuff back in we see that $AB : BC : CA = 4 : 4 : 6$. An isosceles triangle with side lengths 4, 4, 6 has area $3\sqrt{7}$ and inradius $\frac{3}{\sqrt{7}}$, so by scaling we see that the area of ABC is $3\sqrt{7} \cdot \left(\frac{3}{3/\sqrt{7}}\right)^2 = 21\sqrt{7} \implies \boxed{3087}$.

G24. Let a and b be positive real numbers, and suppose that the graphs of the equations

$$y = ax^2 - b \quad \text{and} \quad x = ay^2 - b$$

intersect at 4 distinct points A , B , C , and D , with their x -coordinates increasing in that order. The perpendicular bisectors of AB and CD meet at P . If the distance from P to the origin is 400, find the smallest possible value of $\lfloor b \rfloor$.

Proposed by: Oliver Zhang

Answer: $\boxed{424}$

Solution: Firstly, observe that the two graphs are reflections over the line $y = x$, and hence by symmetry, B and D lie on $y = x$, and hence their x and y -coordinates are the roots of $x = ax^2 - b$.

Claim. $ABCD$ is cyclic.

Proof. Suppose that (x, y) is a point of intersection of the two graphs. We will show that (x, y) lies on a fixed circle.

Indeed, by summing up the two equations and rearranging, we get

$$\begin{aligned}x + y &= ax^2 + ay^2 - 2b \\ax^2 - x + ay^2 - y &= 2b \\x^2 - \frac{1}{a}x + y^2 - \frac{1}{a}y &= 2\frac{b}{a} \\(x - \frac{1}{2a})^2 + (y - \frac{1}{2a})^2 &= 2\frac{b}{a} + \frac{1}{2a^2},\end{aligned}$$

which is the equation of a fixed circle. $\square$

By the claim above, P is the center of $(ABCD)$. Furthermore, since $(ABCD)$ is centered at $(\frac{1}{2a}, \frac{1}{2a})$, we have $\frac{\sqrt{2}}{2a} = 400 \implies a = \frac{\sqrt{2}}{800}$.

Now, to finish, we must find where A and C lie. Plugging in $y = ax^2 - b$ into $x = ay^2 - b$, we get the quartic that yields the solutions for x :

$$x = a(ax^2 - b)^2 - b \implies a^3x^4 - 2a^2bx^2 - x + ab^2 - b = 0.$$

Since two of the solutions, B and D , satisfy $x = ax^2 - b$, we may factor out $(ax^2 - x - b)$ to get

$$(ax^2 - x - b)(a^2x^2 + ax + 1 - ab) = 0.$$

The right quantity has solutions

$$x = \frac{-a \pm \sqrt{a^2 + 4a^2(ab - 1)}}{2a^2} = \frac{-1 \pm \sqrt{4ab - 3}}{2a},$$

which only exist when $4ab - 3 \geq 0 \iff ab \geq \frac{3}{4}$. Furthermore, since A and C must be distinct, we actually need $ab > \frac{3}{4}$, or $b > \frac{3}{4a}$. Plugging in the value for a we have

$$b > \frac{3}{4} \cdot \frac{800}{\sqrt{2}} = \frac{3 \cdot 800\sqrt{2}}{8} = 300\sqrt{2}.$$

Since $1.414 < \sqrt{2} < 1.415$, we have $424.2 < 300\sqrt{2} < 424.5$, implying $\lfloor b \rfloor > \lfloor 424.2 \rfloor = \lfloor 424.5 \rfloor = \boxed{424}$.

- G25. Estimate N , the total number of participants (in person and online) at MOAA this year.

Answer: $\boxed{347}$

- G26. Estimate L , the total number of letters in all the teams that signed up for MOAA this year, both in person and online.

Answer: $\boxed{1829}$

- G27. Estimate S , the sum of all submitted answers across every round of MOAA 2026 (besides Gunga Bowl Problem 1 and Set 9) that includes either the digit 6 or 7.

Answer: $\boxed{78451334}$

Note: non-integer submissions were omitted.