

MOAA 2026 Team Round Solutions

MATH OPEN AT ANDOVER

October 10th, 2026

T1. Compute

$$\sqrt{(2026^2 - 1^2)(2026 + 1)}.$$

Proposed by: Paige Zhu

Answer: $\boxed{91215}$

Solution: Since $\sqrt{2025} = 45$, we have

$$\sqrt{(2026^2 - 1^2)(2026 + 1)} = \sqrt{(2026 + 1)(2026 - 1)(2026 + 1)} = 2027 \cdot 45 = \boxed{91215}.$$

T2. The numbers 1, 2, ..., 14 are to be entered into the 14 boxes shown below, so that each number is used exactly once:

$$P = (\square + \square + \square + \square + \square + \square + \square)(\square + \square + \square + \square + \square + \square + \square).$$

Find the maximum value of P .

Proposed by: Athreya Yegneswaran

Answer: $\boxed{2756}$

Solution: Firstly, $1 + 2 + \cdots + 14 = \frac{1}{2} \cdot 14 \cdot 15 = 105$. Suppose the two parenthesis have sums X and Y , respectively. Since

$$P = XY = \frac{1}{4} ((X + Y)^2 - (X - Y)^2)$$

and $X + Y$ is fixed, we see that XY is maximized when $X - Y$ is minimized. Also, both X and Y are integers, and $X + Y$ is odd, so $X - Y$ is at least 1, achieved with $X = 52$ and $Y = 53$. Then $XY = \boxed{2756}$. Indeed, this is achievable:

$$P = 52 \cdot 53 = (14 + 13 + 12 + 5 + 4 + 3 + 1)(11 + 10 + 9 + 8 + 7 + 6 + 2) = 2756.$$

T3. Let a and b be nonzero real numbers such that $(a - b)^2 = 3ab$. If $\left(\frac{a+b}{a-b}\right)^4$ can be written as $\frac{m}{n}$ for relative prime positive integers m and n , find $m + n$.

Proposed by: Paige Zhu

Answer: $\boxed{58}$

Solution: From $(a - b)^2 = 3ab$ we get $a^2 + b^2 = 5ab$. Then

$$\left(\frac{a+b}{a-b}\right)^4 = \left(\frac{(a+b)^2}{(a-b)^2}\right)^2 = \left(\frac{a^2+b^2+2ab}{a^2+b^2-2ab}\right)^2 = \left(\frac{7ab}{3ab}\right)^2 = \frac{49}{9} \implies \boxed{58}.$$

- T4. Paige is reviewing a printed manuscript for the LIVE textbook. She notices that the sum of the page numbers on 26 consecutive pages is a perfect square. Find the smallest possible page number of the first page she looks at.

Proposed by: Paige Zhu

Answer: $\boxed{46}$

Solution: Let the first page have number n . Then the sum of those 26 consecutive pages is

$$n + (n + 1) + \cdots + (n + 25) = 26n + (1 + \cdots + 25) = 26n + 25 \cdot 13$$

Suppose $26n + 25 \cdot 13 = N^2$. Since the left side is divisible by 13, which is prime, we have $13 \mid N^2 \implies 13 \mid N$, so we can set $N = 13m$. Then

$$\begin{aligned} 26n + 25 \cdot 13 &= 13^2 m^2 \implies 2n + 25 = 13m^2 \\ \implies 2n + 12 &= 13(m^2 - 1) \implies 2(n + 6) = 13(m + 1)(m - 1). \end{aligned}$$

We need $(m + 1)(m - 1)$ to be even, so m must be odd. $m = 1$ implies n is negative, which is absurd, so the smallest value of m is 3, where we get

$$2(n + 6) = 13 \cdot 4 \cdot 2 \implies n + 6 = 52 \implies n = \boxed{46}.$$

- T5. Let ABC be a triangle with $AB = 11$, $BC = 9$, and $CA = 7$. Let A' be the reflection of A across the altitude from B to AC . If the circumcircle of $\triangle A'CB$ intersects the line AB again at D , the length of BD can be written as $\frac{m}{n}$ for relatively prime positive integers m and n . Find $m + n$.

Proposed by: Carolyn Cao

Answer: $\boxed{43}$

Solution: The semiperimeter of ABC is $s = \frac{27}{2}$. Then by Heron's formula,

$$[ABC] = \sqrt{\frac{27}{2} \cdot \frac{5}{2} \cdot \frac{13}{2} \cdot \frac{9}{2}} = \frac{9}{4} \sqrt{195}.$$

Let X be the foot from B to AC . We have

$$BX = \frac{2[ABC]}{AC} = \frac{9}{14} \sqrt{195}.$$

By the Pythagorean theorem,

$$\begin{aligned} AX &= \sqrt{AB^2 - BX^2} \\ &= \sqrt{121 - \frac{81 \cdot 195}{196}} \\ &= \sqrt{\frac{196 \cdot 121 - 81 \cdot 195}{196}} \\ &= \sqrt{\frac{23716 - 13795}{196}} \\ &= \sqrt{\frac{7921}{196}} = \frac{89}{14}. \end{aligned}$$

Then, by power of a point from A to the circumcircle of $A'CB$, we get

$$AD \cdot AB = AC \cdot AA' \implies AD = \frac{AC \cdot 2AX}{11} = \frac{1}{11} \cdot 7 \cdot 2 \cdot \frac{89}{14} = \frac{89}{11}.$$

Finally,

$$BD = 11 - AD = 11 - \frac{89}{11} = \frac{32}{11} \implies 32 + 11 = \boxed{43}.$$

T6. Find the sum of all values of n satisfying

$$n - \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{n}{8} \right\rfloor = 2026.$$

Proposed by: Ishan Gandhi

Answer: $\boxed{6483}$

Solution: Firstly, since the three floors and the number 2026 are all integers, n must be an integer. Now, we do casework on the remainder of n (mod 8):

- $n = 8k + 1$: $8k + 1 - 4k + 2k - k = 2026 \implies k = 405$, so $n = 3241$.
- $n = 8k + 2$: $8k + 2 - 4k - 1 + 2k - k = 2026 \implies k = 405$, so $n = 3242$.
- $n = 8k + 3$: $8k + 3 - 4k - 1 + 2k - k = 2026 \implies 5k = 2024$, impossible.
- $n = 8k + 4$: $8k + 4 - 4k - 2 + 2k + 1 - k = 2026 \implies 5k = 2024$, impossible.

The other 4 cases result in similar contradictions, so the only two possible values for n are 3241 and 3242. Then $3241 + 3242 = \boxed{6483}$.

T7. Find the number of 7-digit numbers can be formed using each of the digits 1 through 7 exactly once such that the sum of any two adjacent digits is a prime number.

Proposed by: Paige Zhu

Answer: $\boxed{24}$

Solution: Firstly, 2 cannot appear as the sum of two adjacent digits because the only case is $1 + 1 = 2$, which is impossible as 1 can only appear once. Hence, any desired number must have alternating even and odd digits. We list out the possible neighboring digits for all odd digits:

- 1 – 2, 4, 6
- 3 – 2, 4
- 5 – 2, 6
- 7 – 4, 6

We claim that each possible placement of the odd digits results in a unique valid 7-digit number.

Since the digits 3, 5, and 7 all have exactly 2 valid neighboring digits, let us call them *special*. Let this 7-digit number be $\overline{AXBYCZD}$, where A, B, C, D are odd and X, Y, Z are even. At least one of the digits B or C will be special. WLOG, let B be special, and let its two valid neighboring digits (hereinafter just called neighbors) be p and q . We have 3 cases:

- A and C are special and $D = 1$. Since any two special numbers share exactly one valid neighbor, there is a unique way of assigning p and q to X and Y such that X and Y are valid neighbors of A and C , respectively. Since the last unused even digit is not a valid neighbor of B , it must be a valid neighbor of C , so assign it to Z and we have constructed a unique example.

- A and D are special and $C = 1$. Again, exactly one of p, q can be assigned to X . Assigning the other one to Y , we end up in a similar situation where the unused even digit can be assigned to Z because it must be a valid neighbor of the special digit D .
- C and D are special and $A = 1$. Similarly, we assign X and Y , then the remaining even digit can be assigned to Z .

Therefore, there are $4! = \boxed{24}$ such numbers.

- T8. Let ABC be an equilateral triangle with side length 10. Points D, E , and F are on sides BC, CA , and AB , respectively, such that $AE = CD = BF = k$. The three segments AD, BE , and CF bound a triangle Δ . Suppose that the incircle of ABC is the circumcircle of Δ . Then $k = a - \sqrt{b}$, where a and b are integers and b is square-free. Find $10a + b$.

Proposed by: Oliver Zhang

Answer: $\boxed{55}$

Solution: Let γ be the incircle of ABC . It is easy to see that γ has radius $\frac{5}{\sqrt{3}}$, and that any equilateral triangle inscribed in γ has side length 5.

Let M be the midpoint of BC , and let AM intersect γ again at N . Let AD intersect γ at X and Y , with $AX < AY$. $AM = 5\sqrt{3}$, so $AN = 5\sqrt{3} - \frac{10}{\sqrt{3}}$. Also, $XY = 5$ since it is the side of an equilateral triangle inscribed in γ . By Power of a Point,

$$AX \cdot AY = AN \cdot AM \implies AX(AX + 5) = \left(5\sqrt{3} - \frac{10}{\sqrt{3}}\right) \cdot 3\sqrt{3} \implies AX = \frac{5\sqrt{5} - 5}{2}.$$

Now, let P be the midpoint of XY and O be the center of γ . Then $AP = AX + \frac{1}{2}XY = \frac{5\sqrt{5} - 5}{2} + \frac{5}{2} = \frac{5\sqrt{5}}{2}$. Also, by $XP = \frac{5}{2}$ we have $OP = \frac{5}{2\sqrt{3}}$. Furthermore, $OP \perp AD$, so $\triangle APO \sim \triangle AMD$ and we get

$$AP : OP = AM : MD \implies MD = \frac{OP \cdot AM}{AP} = \frac{\frac{5}{2\sqrt{3}} \cdot 5\sqrt{3}}{\frac{5\sqrt{5}}{2}} = \frac{\frac{25}{2}}{\frac{5\sqrt{5}}{2}} = \sqrt{5}.$$

Finally, $k = CD = MC - MD = 5 - \sqrt{5} \implies \boxed{55}$.

- T9. Let k be the largest real number such that the inequality

$$\sqrt{\frac{x^2 + y^2}{2}} - k \geq \frac{2}{\frac{1}{x} + \frac{1}{y}}$$

holds for all positive integers x and y with $x \neq y$. Then $k = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Oliver Zhang

Answer: $\boxed{31}$

Solution: WLOG let $y > x$, and make the substitution $y = x + a$, where $x, a \geq 1$. Rewrite the inequality as

$$\sqrt{\frac{x^2 + y^2}{2}} - k \geq \frac{2xy}{x + y} \implies k \leq \frac{x^2 + y^2}{2} - \frac{4x^2y^2}{(x + y)^2} = \frac{x^2 + (x + a)^2}{2} - \frac{4x^2(x + a)^2}{(2x + a)^2}.$$

Now we simplify:

$$\begin{aligned}
 \frac{x^2 + (x+a)^2}{2} - \frac{4x^2(x+a)^2}{(2x+a)^2} &= \frac{(x+a)^2 + x^2}{2} - \frac{4x^4 + 8x^3a + 4x^2a^2}{(2x+a)^2} \\
 &= \frac{(x+a)^2 + x^2}{2} - \frac{x^2(4x^2 + 4xa + a^2) + x^2a(4x + 3a)}{(2x+a)^2} \\
 &= \frac{(x+a)^2 + x^2}{2} - x^2 - \frac{x^2a(4x + 3a)}{(2x+a)^2} \\
 &= \frac{a(2x+a)}{2} - \frac{x^2a(4x + 3a)}{(2x+a)^2} \\
 &= \frac{a}{2} \cdot \frac{8x^3 + 12x^2a + 6xa^2 + a^3 - 8x^3 - 6x^2a}{(2x+a)^2} \\
 &= \frac{a}{2} \cdot \frac{6x^2a + 6xa^2 + a^3}{(2x+a)^2}.
 \end{aligned}$$

We need k to be at most the expression above as x and a range over the positive integers. Observe that every term in the expression above is positive, so this expression strictly increases as x or a increase, and therefore has its minimum at $x = a = 1$, where it evaluates to $\frac{13}{18} \implies 13 + 18 = \boxed{31}$.

T10. Let $f(x) = \frac{4^x}{4^x+2}$. Evaluate

$$2 \sum_{k=1}^{2025} f\left(\frac{k}{2026}\right).$$

Proposed by: Carolyn Cao

Answer: $\boxed{2025}$

Solution: Notice that evaluating $f(x) + f(1-x)$ yields

$$\begin{aligned}
 f(1-x) &= \frac{4^{1-x}}{4^{1-x}+2} = \frac{\frac{4}{4^x}}{\frac{4}{4^x}+2} = \frac{4}{4+2 \cdot 4^x} = \frac{2}{2+4^x} \\
 f(x) + f(1-x) &= \frac{4^x}{4^x+2} + \frac{2}{4^x+2} = 1
 \end{aligned}$$

Therefore, we pair up the elements of our sum from opposite ends

$$f\left(\frac{k}{2026}\right) + f\left(\frac{2026-k}{2026}\right) = 1$$

There are 2025 terms total. This creates $\frac{2025-1}{2} = 1012$ pairs that sum to 1, meaning the only unpaired term is the middle term, which is

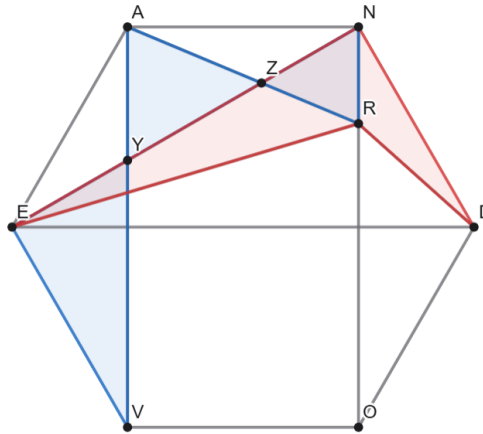
$$\begin{aligned}
 f\left(\frac{1013}{2026}\right) &= f\left(\frac{1}{2}\right) = \frac{4^{1/2}}{4^{1/2}+2} = \frac{2}{2+2} = \frac{1}{2} \\
 \implies \text{total sum} &= 2 \left(1012 \cdot 1 + \frac{1}{2}\right) = \boxed{2025}.
 \end{aligned}$$

- T11. Let $ANDOVE$ be a regular hexagon, and let R be a point on segment $\overline{ON}$. Suppose that the quadrilateral $NERD$ has area 20 and the self-intersecting pentagon $RAVEN$ has area 26. Then the area of quadrilateral $DOVE$ can be written in the form $a + b\sqrt{c}$, where a, b, c are positive integers and c is square-free. Find $100a + 10b + c$.

Proposed by: Oliver Zhang

Answer: 1986

Solution:



Let $ANDOVE$ have center X and side length a , and let $\overline{EN}$ intersect $\overline{AV}$ and $\overline{AR}$ at Y and Z , respectively. Let $NR = h$. Since $NERD$ has perpendicular diagonals, $[NERD] = \frac{1}{2}NR \cdot ED = ah = 20$. The rest is straightforward computations.

Since Y is independent of R , so is $[EYV]$, so some simple length chase with $30-60-90$ triangles yields $AY = \frac{\sqrt{3}}{3}a$ and $[EYV] = \frac{\sqrt{3}}{6}a^2$.

Now, let k and k' be the heights from Z to $\overline{AY}$ and $\overline{NR}$, respectively. Then $k + k' = a$. Furthermore, since $\triangle AZY \sim \triangle RZN$, we get

$$k : a - k = AY : NR \implies k : a - k = \frac{\sqrt{3}}{3}a : h \implies k = \frac{\frac{\sqrt{3}}{3}a^2}{\frac{\sqrt{3}}{3}a + h} = \frac{a^2}{a + \sqrt{3}h}.$$

Therefore, we have $k' = a - k = \frac{\sqrt{3}ah}{a + \sqrt{3}h}$, so

$$[AYZ] + [RNZ] = \frac{1}{2} \cdot \frac{\sqrt{3}}{3}a \cdot \frac{a^2}{a + \sqrt{3}h} + \frac{1}{2} \cdot h \cdot \frac{\sqrt{3}ah}{a + \sqrt{3}h} = \frac{1}{2} \cdot \frac{1}{a + \sqrt{3}h} \left(\frac{\sqrt{3}}{3}a^3 + \sqrt{3}ah^2 \right),$$

and

$$[RAVEN] = \frac{\sqrt{3}}{6}a^2 + \frac{1}{2} \cdot \frac{1}{a + \sqrt{3}h} \left(\frac{\sqrt{3}}{3}a^3 + \sqrt{3}ah^2 \right) = 26.$$

We proceed to simplify this expression:

$$\begin{aligned} \frac{\sqrt{3}}{6}a^2 (a + \sqrt{3}h) + \frac{1}{2} \left(\frac{\sqrt{3}}{3}a^3 + \sqrt{3}ah^2 \right) &= 26 (a + \sqrt{3}h) \\ \frac{\sqrt{3}}{3}a^3 + \frac{1}{2}a^2h + \frac{\sqrt{3}}{2}ah^2 &= 26 (a + \sqrt{3}h) \end{aligned}$$

Observe that, since $ah = 20$,

$$\frac{1}{2}a^2h + \frac{\sqrt{3}}{2}ah^2 = \frac{1}{2}ah(a + \sqrt{3}h) = 10(a + \sqrt{3}h),$$

so the above simplifies further to

$$\frac{\sqrt{3}}{3}a^3 = 16(a + \sqrt{3}h) \implies a^3 - 16\sqrt{3}a - 48h = 0.$$

Since $[DOVE]$ only depends on the value of a^2 , we can substitute $h = \frac{20}{a}$ and multiply both sides by a to get a quadratic with a^2 as the unknown:

$$\begin{aligned} a^3 - 16\sqrt{3}a - \frac{960}{a} &= 0 \\ a^4 - 16\sqrt{3}a^2 - 960 &= 0 \\ a^2 &= \frac{16\sqrt{3} \pm \sqrt{4608}}{2} = 8\sqrt{3} \pm 24\sqrt{2}. \end{aligned}$$

Since $24\sqrt{2} > 8\sqrt{3}$, obviously we need $a^2 = 8\sqrt{3} + 24\sqrt{2}$. Then

$$[DOVE] = 3 \cdot \frac{\sqrt{3}}{4}a^2 = 3\sqrt{3} \cdot (2\sqrt{3} + 6\sqrt{2}) = 18 + 18\sqrt{6} \implies 1800 + 180 + 6 = \boxed{1986}.$$

- T12. Call a pair of positive integers (a, b) *compatible* if a divides $12b + 1$ and b divides $12a + 1$. Compute the sum of $(a + b)$ across all compatible pairs (a, b) for which $a \geq b \geq 2$.

Proposed by: Oliver Zhang

Answer: $\boxed{228}$

Solution: Let (a, b) be a compatible pair.

Claim. $\gcd(a, b) = 1$.

Proof. Suppose that $\gcd(a, b) = d > 1$. Then since a divides $12b + 1$ we must have that d also divides $12b + 1$, but since d divides b and $d > 1$, this is impossible. $\square$

Therefore, let $ma = 12b + 1$, where m is an integer and $\gcd(m, 12) = 1$ by the claim above. Then $a = \frac{12}{m}b + \frac{1}{m}$. Similarly, $b = \frac{12}{n}a + \frac{1}{n}$ for some integer n with $\gcd(n, 12) = 1$. Solving for the pair (a, b) in terms of m and n , we get

$$b = \frac{12}{n} \left(\frac{12}{m}b + \frac{1}{m} \right) + \frac{1}{n} \implies mnb = 144b + 12 + m \implies b = \frac{m + 12}{mn - 144},$$

so

$$a = \frac{1}{m}(12b + 1) = \frac{1}{m} \cdot \frac{144 + 12m + mn - 144}{mn - 144} = \frac{n + 12}{mn - 144}.$$

Therefore, we need $mn - 144$ to be positive and divide both $m + 12$ and $n + 12$.

Since $a \geq b$ and $a = \frac{12}{m}b + \frac{1}{m}$ where $\frac{1}{m} \leq 1$, we only have to consider $m \leq 12$, so the only four choices of m we have are 1, 5, 7, and 11. In each of the four cases we see that $mn - 144$ must divide $m + 12 = 13, 17, 19$, and 23 , respectively, all of which are prime. Therefore, $mn - 144$ is either 1 or $m + 12$.

Since $mn - 144 = m + 12$ implies $b = 1 < 2$, which is not allowed, we must have $mn - 144 = 1$ always, so $mn = 145$. $145 = 5 \cdot 29$, so since $n = \frac{145}{m}$, the only two m that work are 1 and 5, yielding $n = 145$ and 29 , respectively. Then $(a, b) = (13, 157)$ or $(17, 41)$, and $13 + 157 + 17 + 41 = \boxed{228}$.

- T13. Let $\mathcal{G}$ be a square lattice of dimension 20×26 consisting of $20 \times 26 = 520$ lattice points forming $19 \times 25 = 475$ unit squares. In each unit square, Oliver randomly draws in either one of the two diagonals, both diagonals, or nothing at all, with each of the 4 cases occurring with equal probability. Let N be the expected number of *finite* regions bounded by the segments Oliver drew. Find the integer closest to N .

Proposed by: Oliver Zhang

Answer: 125

Solution: Replace 20 and 26 with $m + 1$ and $n + 1$, respectively. The idea is to use linearity of expectation on Euler's formula $F = E - V + 2$. In this case, we discard the outer infinite region to get $F = E - V + 1$. Then taking linearity gives $\mathbb{E}(F) = \mathbb{E}(E) - \mathbb{E}(V) + 1$.

Now, to find $\mathbb{E}(E)$, we focus on each individual square. In each square, there is an equal probability of there being 0, 1, 1, or 4 segments, corresponding to no diagonals, one of the two diagonals, or both diagonals, respectively, so $\mathbb{E}(E) = mn \cdot \frac{1+1+4}{4} = \frac{3}{2}mn$.

For $\mathbb{E}(V)$, we first consider each lattice point individually.

- The 4 lattice points in the corner each have a $\frac{1}{2}$ chance of being connected to at least one segment, adding $4 \cdot \frac{1}{2} = 2$ to the total.
- The $2(m - 1) + 2(n - 1) = 2m + 2n - 4$ lattice points on the edge each have a $1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$ chance of being connected to at least one segment (by complimentary counting), adding $\frac{3}{2}(m + n - 2)$ to the total.
- The $(m - 1)(n - 1)$ lattice points in the center each have a $1 - \left(\frac{1}{2}\right)^4 = \frac{15}{16}$ chance of being connected to at least one segment (complimentary counting again), adding $\frac{15}{16}(mn - m - n + 1)$ to the total.

Furthermore, in each square, there is a $\frac{1}{4}$ chance of there being an extra segment from the two intersecting diagonals, adding $\frac{1}{4}mn$ to the count. Therefore,

$$\mathbb{E}(V) = 2 + \frac{3}{2}(m + n - 2) + \frac{15}{16}(mn - m - n + 1) + \frac{1}{4}mn = \frac{19}{16}mn + \frac{9}{16}(m + n) - \frac{1}{16}.$$

Finally,

$$\mathbb{E}(F) = \frac{3}{2}mn - \frac{19}{16}mn - \frac{9}{16}(m + n) + \frac{1}{16} + 1 = \frac{1}{16}(5mn - 9(m + n) + 17).$$

However, we are not quite done: if there are no segments drawn, Euler's formula gives 1 region, which is false. 1 region contributes $\frac{1}{4^{mn}}$ to the expected value, so we need to subtract that to yield

$$\mathbb{E}(F) = \frac{1}{16}(5mn - 9(m + n) + 17) - \frac{1}{4^{mn}}.$$

Evaluating this for $m = 19$ and $n = 25$ gives

$$\mathbb{E}(F) = \frac{1}{16}(1979 - 396 + 17) - \frac{1}{4^{475}} = \frac{1996}{16} - \frac{1}{4^{475}} = 124.75 - \frac{1}{4^{475}} \approx \boxed{125}.$$

- T14. Let T_n be the number of ways to color some squares in a $3 \times n$ grid black such that no two black squares are adjacent. As n grows large, the ratio $\frac{T_{n+1}}{T_n}$ approaches some real number L . Let f be the unique monic polynomial with integer coefficients of lowest degree such that $f(L) = 0$.

(Clarification in test: diagonal squares are NOT considered adjacent, and coloring no squares is allowed.)

Proposed by: Oliver Zhang and Pranav Mallina

Answer: 7401

Solution: Let $X_{ijk}(n)$, where $i, j, k \in \{0, 1\}$, denote the number of ways to color some squares in a $3 \times n$ grid black such that no two black squares are adjacent, and that the rightmost column of 3 grids are colored i, j, k , respectively, with 1 = black and 0 = white. We have $X_{000}(1) = X_{001}(1) = X_{010}(1) = X_{100}(1) = X_{101}(1) = 1$. We proceed by recursion:

$$\begin{cases} X_{000}(n+1) = X_{000}(n) + X_{001}(n) + X_{010}(n) + X_{100}(n) + X_{101}(n) \\ X_{001}(n+1) = X_{000}(n) + X_{010}(n) + X_{100}(n) \\ X_{100}(n+1) = X_{000}(n) + X_{010}(n) + X_{001}(n) \\ X_{010}(n+1) = X_{000}(n) + X_{001}(n) + X_{100}(n) + X_{101}(n) \\ X_{101}(n+1) = X_{000}(n) + X_{010}(n) \end{cases},$$

We know $T_n = X_{000}(n) + X_{001}(n) + X_{100}(n) + X_{010}(n) + X_{101}(n)$, but that is just equal to $X_{000}(n+1)$. Hence it suffices to find a recursion for X_{000} .

Observe that, by symmetry, $T_{001}(n) = T_{100}(n)$. Now, for the sake of simplicity, let $A(n) = T_{000}(n)$, $B(n) = T_{001}(n) = T_{100}(n)$, $C(n) = T_{010}(n)$, and $D(n) = T_{101}(n)$. Then the recursion system becomes slightly cleaner:

$$\begin{cases} A(n) = A(n-1) + 2B(n-1) + C(n-1) + D(n-1) \\ B(n) = A(n-1) + B(n-1) + C(n-1) \\ C(n) = A(n-1) + 2B(n-1) + D(n-1) \\ D(n) = A(n-1) + C(n-1) \end{cases}$$

Now, we can do some manipulation on $A(n)$ to get it in terms of only $A(k)$ for $k < n$:

$$\begin{aligned} A(n) &= A(n-1) + 2B(n-1) + C(n-1) + D(n-1) \\ &= A(n-1) + 4A(n-2) + 4B(n-2) + 3C(n-2) + D(n-2) \\ &= 2A(n-1) + 3A(n-2) + 2B(n-2) + 2C(n-2) \\ &= 2A(n-1) + 3A(n-2) + 4A(n-3) + 6B(n-3) + 2C(n-3) + 2D(n-3) \\ &= 2A(n-1) + 6A(n-2) + A(n-3) - C(n-3) - D(n-3). \end{aligned}$$

But notice that

$$\begin{aligned} A(n-3) - C(n-3) - D(n-3) &= A(n-4) + 2B(n-4) + C(n-4) + D(n-4) \\ &\quad - A(n-4) - 2B(n-4) - D(n-4) \\ &\quad - A(n-4) - C(n-4) \\ &= -A(n-4), \end{aligned}$$

so we have

$$A(n) = 2A(n-1) + 6A(n-2) - A(n-4)$$

for all $n > 4$. Therefore, this is also true for T_n for $n > 5$. We then have

$$\begin{aligned} T_n &= 2T_{n-1} + 6T_{n-2} - T_{n-4} \\ \implies \frac{T_n}{T_{n-4}} &= 2\frac{T_{n-1}}{T_{n-4}} + 6\frac{T_{n-2}}{T_{n-4}} - 1 \\ \implies \frac{T_n}{T_{n-4}} - 2\frac{T_{n-1}}{T_{n-4}} - 6\frac{T_{n-2}}{T_{n-4}} + 1 &= 0. \end{aligned}$$

We are given that $\frac{T_{n+1}}{T_n}$ becomes constant as n grows large. Therefore, $\frac{T_n}{T_{n-4}}$ approaches L^4 , and similarly for the other terms, so we get

$$L^4 - 2L^3 - 6L^2 + 1 = 0,$$

and hence $f(x) = x^4 - 2x^3 - 6x^2 + 1$ is a desired polynomial f . By the rational roots theorem we can quickly see that f has no factors of the form $ax + b$, where a and b are integers, and one can check easily that it doesn't have integer-coefficient quadratic factors either. Since f is also monic, it must be the desired polynomial with lowest degree. Then $f(10) = 10000 - 2000 - 600 + 1 = \boxed{7401}$.

- T15. Let ABC be a triangle with $AC = 3AB$. Let H be the orthocenter of ABC and M be the midpoint of BC . A point P is located inside of $\triangle ABC$ such that $\angle ABP = \angle ACP$. If $AM^2 = 10$ and the smallest possible value of PM^2 is 6, find the sum of all possible values of $\lfloor HM^2 \rfloor$.

Note: $\lfloor n \rfloor$ denotes the largest integer less than or equal to n .

Proposed by: Oliver Zhang

Answer: $\boxed{1302}$

Solution: We start with defining the point P in a slightly different way: let X be a point on line AB , and suppose the circumcircle of BCX intersects line AC at Y . Then P is the intersection of lines CX and BY . It is easy to check that this definition is equivalent with the problem statement. Furthermore, this allows us to consider points P outside of $\triangle ABC$.

Claim. The set of all points P satisfying the condition is

- A hyperbola $\mathcal{H}$ passing through A , B , and C , and H , with
- center M , and
- perpendicular asymptotes.

Proof. We first show that $\mathcal{H}$ is a hyperbola. Let $AX = t$. By power of a point on circle $BXYC$ we see that AY is linear in t . Since CX and BY are two lines with one degree of freedom, their intersection must have 2 degrees of freedom, hence $\mathcal{H}$ is a conic.

Now, consider the following:

- Choosing $X = A$ or B forces $P = A$ or B , respectively.
- Choosing a point X such that the circumcircle of BCX is tangent to line AC forces $P = C$.

- If N and M are the feet from B and C to CA and AB , respectively, then $BNMC$ is cyclic and H is the intersection of BN and CM , so in this case $P = H$.

Finally, since P takes on points inside ABC , it must move along some hyperbola $\mathcal{H}$.

Now, pick any point P , and let P' be the reflection of P over M . Note that $PCP'B$ is a parallelogram. Therefore,

$$\angle ABP' = \angle ABP + \angle PBP' = \angle ACP + \angle PCP' = \angle ACP',$$

so P' is also a valid point. Therefore, M must be the center of $\mathcal{H}$.

To find the asymptotes of $\mathcal{H}$, let ℓ be the angle bisector of $\angle A$. Firstly, let ℓ_B and ℓ_C be the lines through B and C , respectively, perpendicular to ℓ . Let X be the intersection of ℓ_C with AB , and Y be the intersection of ℓ_B with AC . Since $BYCX$ is an isosceles trapezoid, it is cyclic. Note that, M lies on the midline of $BYCX$ which is parallel to BY and CX , so one asymptote of $\mathcal{H}$ is parallel to ℓ_B and ℓ_C , hence perpendicular to ℓ .

Similarly, let m_B and m_C be lines through B and C , respectively, parallel to ℓ . Let X be the intersection of m_C with AB and let Y be the intersection of m_B with AC . Then $BYCX$ is also an isosceles trapezoid and M lies on its midline, so the other asymptote of $\mathcal{H}$ is parallel to ℓ . This finishes the proof. $\square$

Now, it is not too hard to see that all hyperbolas with perpendicular asymptotes can be described by $y = \frac{k}{x}$. Hence, we can toss $\triangle ABC$ onto the coordinate plane with M being the origin. Then by symmetry, we can let $B = (b, \frac{k}{b})$ and $C = (-b, -\frac{k}{b})$. Also, let $A = (a, \frac{k}{a})$ and $H = (h, \frac{k}{h})$.

Firstly, by symmetry again, the smallest value of PM is the smallest distance from $\mathcal{H}$ to the origin, which occurs on the line $y = x$. Since this smallest value is given to be $\sqrt{6}$, we can quickly find that the coordinates of P such that this occurs to be $(\sqrt{3}, \sqrt{3})$, so $k = 3$.

Now, $AM^2 = 10$ implies $a^2 + (\frac{k}{a})^2 = 10$, so since $k^2 = 9$, we have

$$a^2 + \frac{9}{a^2} = 10 \implies a^4 - 10a^2 + 9 = 0 \implies a^2 = 1, 9 \implies a = 1, 3.$$

Finally, we look at the condition $AC = 3AB$. Square both sides to get $AC^2 = 9AB^2$. Writing this in coordinates gives

$$(a+b)^2 + \left(\frac{k}{a} + \frac{k}{b}\right)^2 = 9 \left((a-b)^2 + \left(\frac{k}{a} - \frac{k}{b}\right)^2 \right).$$

Moving the 9 into the squares, shifting terms around, and using difference of squares, this simplifies:

$$\begin{aligned} (a+b)^2 - (3a-3b)^2 &= \left(3\frac{k}{a} - 3\frac{k}{b}\right)^2 - \left(\frac{k}{a} + \frac{k}{b}\right)^2 \\ (4a-2b)(4b-2a) &= \left(4\frac{k}{a} - 2\frac{k}{b}\right) \left(2\frac{k}{a} - 4\frac{k}{b}\right) \\ (2a-b)(2b-a) &= k^2 \left(\frac{2b-a}{ab}\right) \left(\frac{b-2a}{ab}\right) \end{aligned}$$

$$(2a - b)(2b - a) = -\frac{k^2}{a^2b^2}(2a - b)(2b - a)$$

$$\implies (2a - b)(2b - a) \left(1 + \frac{k^2}{a^2b^2}\right) = 0.$$

Since $\frac{k^2}{a^2b^2} \geq 0$, we must have $b = 2a$ or $b = \frac{1}{2}a$.

Finally, H can be seen as the intersection of the perpendicular from A to BC with the hyperbola. Therefore, we only need to consider the equation $AH \perp BC$, or equivalently that their slopes have product -1 :

$$\frac{\frac{k}{h} - \frac{k}{a}}{h - a} \cdot \frac{2\frac{k}{b}}{2b} = -1 \implies k \frac{\frac{a-h}{ah}}{h - a} \cdot \frac{k}{b^2} = -1 \implies k^2 = ahb^2 \implies h = \frac{k^2}{ab^2}.$$

We seek the quantity

$$\lfloor HM^2 \rfloor = \left\lfloor h^2 + \frac{k^2}{h^2} \right\rfloor = \left\lfloor \frac{k^4}{a^2b^4} + \frac{a^2b^4}{k^2} \right\rfloor.$$

Since either $b = 2a$ or $b = \frac{1}{2}a$, we get

$$\lfloor HM^2 \rfloor = \left\lfloor \frac{k^4}{16a^6} + \frac{16a^6}{k^2} \right\rfloor \quad \text{or} \quad \lfloor HM^2 \rfloor = \left\lfloor \frac{16k^4}{a^6} + \frac{a^6}{16k^2} \right\rfloor.$$

Plugging in $(k, a) = (3, 1)$ or $(k, a) = (3, 3)$, we arrive at the two possible values

$$\lfloor HM^2 \rfloor = \left\lfloor 16 \cdot 3^4 + \frac{1}{16 \cdot 3^2} \right\rfloor \quad \text{or} \quad \lfloor HM^2 \rfloor = \left\lfloor \frac{16}{3^2} + \frac{3^4}{16} \right\rfloor.$$

The former is equal to $16 \cdot 3^4 = 1296$, and we can approximate the latter as $1.8 + 5 \approx 6.8$, so its floor is 6. Then the answer is $1296 + 6 = \boxed{1302}$.